Exercise 5.3.2 Show that $\lim_{x\to 4^+} \frac{7}{4-x} = -\infty$ and $\lim_{x\to 4^-} \frac{7}{4-x} = +\infty$.

Solution: Let $M \in (-\infty, 0)$ and choose $\delta = -\frac{7}{M} > 0$. Suppose that $x \in (4, 4+\delta)$, i.e. $4 < x < 4+\delta$. Hence $-4 > -x > -4-\delta$ and we get

$$0 > 4 - x > -\delta = -\left(-\frac{7}{M}\right) = \frac{7}{M}.$$

Therefore, $\frac{1}{4-x} < \frac{M}{7}$ and we obtain $\frac{7}{4-x} < M$. Thus we have shown for any negative M, there is some $\delta > 0$ so that for all $x \in (4, 4+\delta)$, $\frac{7}{4-x} < M$, completing the proof that $\lim_{x\to 4^+} \frac{7}{4-x} = -\infty$. The proof of the other limit works the same way.

Exercise 5.3.3 Verify that $\lim_{x\to+\infty} \frac{x+1}{x+2} = 1$.
Solution:
Thinking: We need to find M so that for all $x \in (M, \infty) \cap D$, $\left|\frac{x+1}{x+2} - 1\right| < \epsilon$. This is equivalent to $\left|\frac{x+1-x-2}{x+2}\right| < \epsilon$, or $\left|\frac{-1}{x+2}\right| < \epsilon$. So we would need $|x+2| < \epsilon$. We can pick M so large that $M > 0$, so $M < x < \infty$ hence $0 < M+2 < x+2 < \infty$. Thus we would have $\frac{1}{x+2} < \epsilon$, i.e. $x + 2 > \frac{1}{\epsilon}$, or $x > \frac{1}{\epsilon} - 2$.
Solution: Let $\epsilon > 0$ and choose $M > \frac{1}{\epsilon} - 2$ and let $M < x < \infty$, so we have $\frac{1}{\epsilon} - 2 < M < x$. Thus $\frac{1}{\epsilon} < x + 2$, hence $\epsilon > \frac{1}{x+2}$. Therefore, now compute for such x,

$$\left|\frac{x+1}{x+2} - 1\right| = \left|\frac{-1}{x+2}\right| = \frac{1}{x+2} < \epsilon,$$

completing the proof. ■

Exercise 5.4.1 Prove: Suppose $D \subset \mathbb{R}$, $\alpha \in \mathbb{R}$, $f\colon D \to \mathbb{R}$, and $g\colon D \to \mathbb{R}$. If f and g are continuous at a, then αf, $f+g$, and fg are continuous at a. Moreover, if $g(x) \neq 0$ for all $x \in D$, then $\frac{f}{g}$ is continuous at a.
Proof: Since f and g are continuous at a, we know that $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$. Using the properties of limits of functions (limit of sum is sum of limits, etc), we see that

$$\lim_{x\to a}(f+g)(x) = \lim_{x\to a} f(x)+g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = f(a)+g(a) = (f+g)(a),$$

completing the proof that $f+g$ is continuous at a. Similarly,

$$\lim_{x\to a}(\alpha f)(x) = \alpha \lim_{x\to a} f(x) = \alpha f(a) = (\alpha f)(a),$$

completing the proof that αf is continuous at a. Similarly,

$$\lim_{x\to a}(fg)(x) = \lim_{x\to a} f(x)g(x) = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = f(a)g(a) = (fg)(a),$$

completing the proof that fg is continuous at a. Finally, compute

$$\lim_{x\to a}\frac{f}{g}(x) = \lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{f(a)}{g(a)} = \left(\frac{f}{g}\right)(a),$$

showing that $\frac{f}{g}$ is continuous at a. ■

Exercise 5.4.3 Show that if $f\colon D \to \mathbb{R}$ is continuous at $a \in D$ and $f(a) > 0$, then there exists an open interval I such that $a \in I$ and $f(x) > 0$ for every $x \in I \cap D$.
Proof: We have that $\frac{f(a)}{2} > 0$. Choose $\epsilon = \frac{f(a)}{2}$. Since f is continuous at a, there exists $\delta > 0$ so that for all $x \in (a-\delta, a+\delta) \cap D$, $|f(x) - f(a)| < \epsilon = \frac{f(a)}{2}$. Therefore we have shown

$$-\frac{f(a)}{2} < f(x) - f(a) < \frac{f(a)}{2}.$$

Add $f(a)$ to get

$$0 < \frac{f(a)}{2} < f(x) < \frac{3f(a)}{2},$$

showing that for any $x \in (a-\delta, a+\delta) =: I$ that $f(x) > 0$, completing the proof. ■

Exercise 5.4.10 Let $D \subset \mathbb{R}$. We say that a function $f\colon D \to \mathbb{R}$ is *Lipschitz* if there exists $\alpha \in \mathbb{R}$, $\alpha > 0$, such that $|f(x) - f(y)| \leq \alpha|x - y|$ for all $x, y \in D$. Show that if f is Lipschitz, then f is continuous.
Solution: Let $\epsilon > 0$ and let $a \in D$. Choose $\delta = \frac{\epsilon}{\alpha}$. If

$$x \in (a-\delta, a+\delta) = \left(a - \frac{\epsilon}{\alpha}, a + \frac{\epsilon}{\alpha}\right),$$

This means that

$$a - \frac{\epsilon}{a} < x < a + \frac{\epsilon}{\alpha},$$

hence

$$-\frac{\epsilon}{\alpha} < x - a < \frac{\epsilon}{\alpha},$$

and we get

$$-\epsilon < \alpha(x-a) < \epsilon,$$

meaning that $\alpha|x-a| < \epsilon$. Therefore, we compute

$$|f(x)-f(a)| \overset{\text{Lipschitz}}{\leq} \alpha|x-a| < \epsilon,$$

completing the proof that f is continuous at a. ■

Exercise 5.4.12 Give an example of a closed interval $[a,b] \subset \mathbb{R}$ and a function $f\colon [a,b] \to \mathbb{R}$ which do not satisfy the conclusion of the Intermediate Value Theorem.

Solution: We will define f to not be continuous to make IVT not hold. For example, let $[a,b] = [0,1]$ and define $f(x) = \begin{cases} -1, & 0 < x < 0.5 \\ 1 & 0.5 \leq x \leq 1 \end{cases}$. In this case $f(0) = -1$ anad $f(1) = 1$ so we can choose $s = 0$ so that $f(0) \leq s \leq f(1)$. But there is no $c \in [a,b]$ so that $f(c) = 0$.