HW12 MTH 427/527 Fall 2024

Exercise 5.1.1 Suppose $D \subset \mathbb{R}$, a is a limit point of D, $f\colon D \to \mathbb{R}$ and $g\colon D \to \mathbb{R}$ if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a} f(x)g(x) = LM$.
Solution: Let $\{x_n\} \in S(D,a)$ be an arbitrary sequence in D that converges to a. Our hypothesis shows that $\lim_{n\to\infty} f(x_n) = L$ and $\lim_{n\to\infty} g(x_n) = M$. Therefore by the limit of a product of sequences theorem, we have

$$\lim_{n\to\infty} f(x_n)g(x_n) = \Big(\lim_{n\to\infty} f(x_n)\Big)\Big(\lim_{n\to\infty} g(x_n)\Big) = LM,$$

and since the sequence $\{x_n\}$ is an arbitrary sequence in $S(D,a)$, we have completed the proof. ■

Exercises 5.1.2 and 5.1.3 are proven the same way!

Exercise 5.1.4 Show that if f is a polynomial and $a \in \mathbb{R}$, then $\lim_{x\to a} f(x) = f(a)$.
Proof: Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$. Using the limit properties of sums and the fact that $\lim_{x\to a} x^n = a^n$ for any $n = 0, 1, 2, \ldots$, we compute

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}\Big(a_0 + \ldots + a_nx^n\Big)\\ &= a_0\Big(\lim_{x\to a} 1\Big) + a_1\Big(\lim_{x\to a} x\Big) + \ldots + a_n\Big(\lim_{x\to a} x^n\Big)\\ &= a_0 + a_1a + \ldots + a_na^n = f(a),\end{aligned}$$

completing the proof. ■

Exercise 5.1.5 Show that if f is a rational function and $a \in \text{dom}(f)$, then $\lim_{x\to a} f(x) = f(a)$.
Proof: Let $f(x) = \dfrac{a_0 + a_1x + \ldots + a_nx^n}{b_0 + b_1x + \ldots + b_mx^m}$. By Exercise 5.1.4 and the fact that limits of quotients are quotients of limits, we obtain

$$\lim_{x\to a} f(x) = \frac{\lim_{x\to a} a_0 + a_1x + \ldots + a_nx^n}{\lim_{x\to a} b_0 + b_1x + \ldots + b_mx^m} = \frac{a_0 + a_1a + \ldots + a_na^n}{b_0 + b_1a + \ldots + b_ma^m} = f(a),$$

completing the proof. ■

Exercise 5.1.8 just appeals to the squeeze theorem for sequences, similar to 5.1.1, 5.1.2, and 5.1.3 above.