HW11 MTH 427/527 Fall 2024

Wednesday, November 6, 2024 10:22 AM

Exercise 4.3.9 Suppose $A_i \subseteq \mathbb{R}$, $i=1,2,\dots,n$ and let $B=\bigcup_{i=1}^{n} A_i$. Show that

$$\overline{B} = \bigcup_{i=1}^{n} \overline{A_i}.$$

Proof: Let $x \in \overline{B}$, meaning that $x \in B \cup B'$, where B' is the set of limit points of B.
If $x \in B$, then clearly $x \in \bigcup_{i=1}^{n} \overline{A_i} = \bigcup_{i=1}^{n}(A_i \cup A_i') = \left(\bigcup_{i=1}^{n} A_i\right) \cup \left(\bigcup_{i=1}^{n} A_i'\right) = B \cup \bigcup_{i=1}^{n} A_i'$.
If $x \in B'$, then x is a limit point of $B=\bigcup_{i=1}^{n} A_i$, meaning that $\forall \epsilon>0$ $(x-\epsilon,x+\epsilon)\cap B\setminus\{x\}$ is nonempty. So let $y \in (x-\epsilon,x+\epsilon)\cap B\setminus\{x\}$. This $y \in B$ and so $y \in \bigcup_{i=1}^{n} A_i$, meaning $y \in A_j$ for some j. So $x \in A_j'$ for some A_j. Thus, $x \in \bigcup_{i=1}^{n} \overline{A_i}$.
Therefore $\overline{B} \subseteq \bigcup_{i=1}^{n} \overline{A_i}$.

Conversely, let $x \in \bigcup_{i=1}^{n} \overline{A_i}$, so $x \in \overline{A_j} = A_j \cup A_j'$ for some j. If $x \in A_j$, then $x \in B = \bigcup_{i=1}^{n} A_i$, so $x \in \overline{B} = B \cup B'$. If $x \in A_j'$, then $\forall \epsilon>0$ $\exists y \in (x-\epsilon,x+\epsilon)\cap A_j\setminus\{x\}$. But this means $\forall \epsilon>0$ $\exists y \in (x-\epsilon,x+\epsilon)\cap B\setminus\{x\}$, so $x \in B'$ and hence $x \in \overline{B}$.

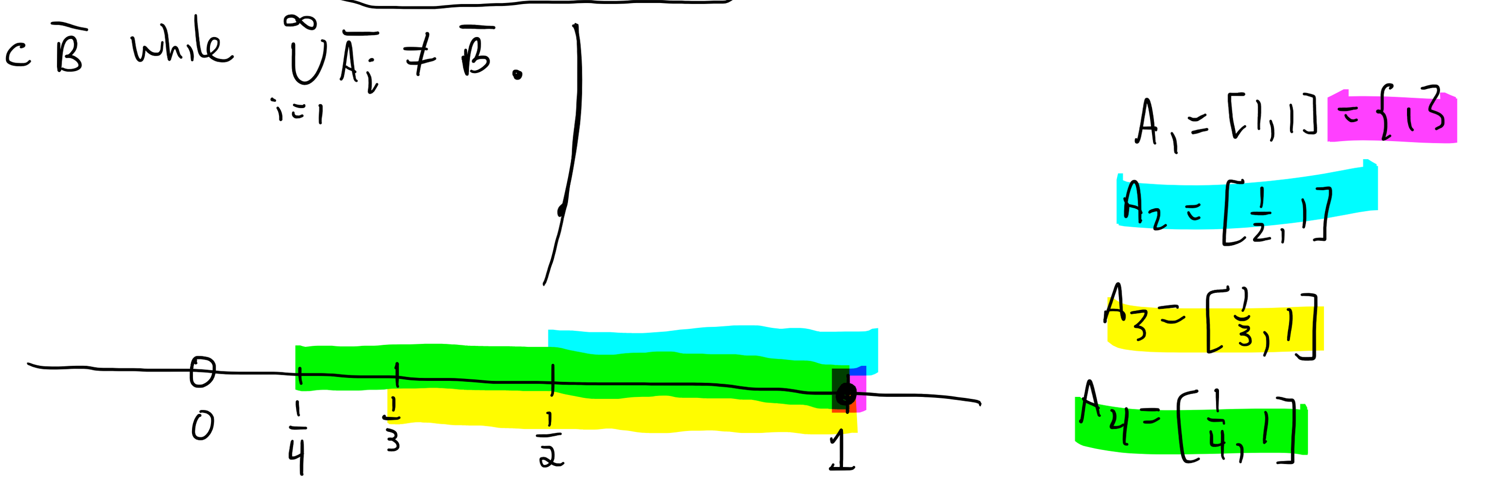
Thus $\overline{B} \subset \bigcup_{i=1}^{n} \overline{A_i}$ and $\bigcup_{i=1}^{n} \overline{A_i} \subseteq \overline{B}$, so $\overline{B} = \bigcup_{i=1}^{n} \overline{A_i}$, completing the proof. ∎

Ex 4.3.10 Suppose $A_i \subseteq \mathbb{R}$ for $i=1,2,\dots$ and let $B=\bigcup_{i=1}^{\infty} A_i$. Show that $\bigcup_{i=1}^{\infty} \overline{A_i} \subset \overline{B}$ and find an example for which $\overline{B} \neq \bigcup_{i=1}^{\infty} \overline{A_i}$.

Soln: **Proof**: Let $x \in \bigcup_{i=1}^{\infty} \overline{A_i}$, so $x \in \overline{A_j}$ for some j, hence $x \in \overline{B} = \bigcup_{i=1}^{\infty} A_i$. Then it follows that $x \in B \cup B' = \overline{B}$, completing the proof. ∎

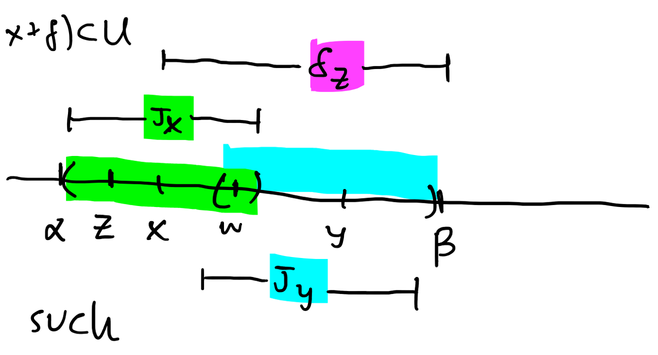
Example: Let $A_i = [\frac{1}{i}, 1] = \overline{A_i}$, so $B = \bigcup_{i=1}^{\infty} [\frac{1}{i}, 1] = (0,1]$, but $\overline{B} = [0,1]$, so we see that $\bigcup_{i=1}^{\infty} \overline{A_i} \subset \overline{B}$ while $\bigcup_{i=1}^{\infty} \overline{A_i} \neq \overline{B}$.

$A_1 = [1,1] = \{1\}$
$A_2 = [\frac{1}{2}, 1]$
$A_3 = [\frac{1}{3}, 1]$
$A_4 = [\frac{1}{4}, 1]$



Ex 4.3.11 Suppose $U \subset \mathbb{R}$ nonempty open set. For each $x \in U$, let $J_x = \bigcup_{\substack{\epsilon,\delta>0 \\ (x-\epsilon,x+\delta)\subset U}} (x-\epsilon, x+\delta)$.

(a) Show that $\forall x,y \in U$, either $J_x \cap J_y = \emptyset$ or $J_x = J_y$.



Proof: If $J_x \cap J_y = \emptyset$, then we are done. So suppose $J_x \cap J_y \neq \emptyset$ and let $w \in J_x \cap J_y$.
Suppose there is some $z \in J_x \setminus J_y$, so $\exists \epsilon_z, \delta_z > 0$ such that $z \in (x-\epsilon_z, x+\delta_z) \subset U$. Since $w \in J_x$, we can pick δ_z so that $w \in (x-\epsilon_z, x+\delta_z)$. But that means δ_z can be picked so that $x+\delta_z = \sup J_x \cup J_y$, so $J_y \subset J_x$. The same argument applied to some $z \in J_y \setminus J_x$ shows $J_x \subset J_y$. Hence $J_x = J_y$, completing the proof. ∎

by choice of ϵ_z

Ex 4.4.2 Show every finite subset of $\mathbb{R}$ is compact.

Proof: Let $X = \{x_1, x_2, \dots, x_n\}$. Let $\theta = \{U_\alpha : \alpha \in A\}$ be an open cover of X, i.e. $X \subset \bigcup \theta$. (some index set)
For each $x_j \in X$ $\exists$ $\theta_j \in \theta$ so that $x_j \in \theta_j$. Then,

$$X \subset \bigcup_{i=1}^{n} \theta_i$$

meaning that $\{\theta_1, \dots, \theta_n\}$ is a finite subcover, completing the proof. ∎

Ex 4.4.3 Suppose $K_1, K_2, \dots, K_n$ are compact sets. Show that $\bigcup_{i=1}^{n} K_i$ is compact.

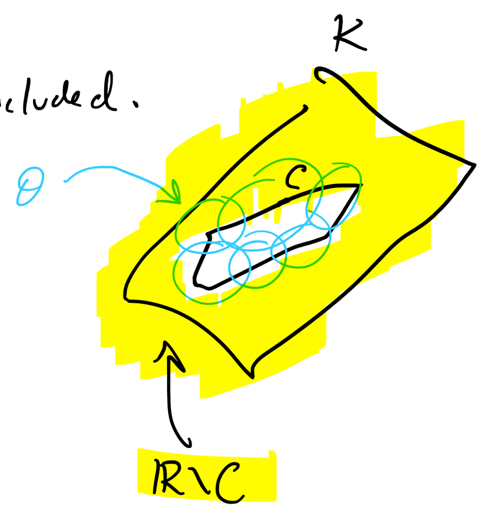
Proof: Let θ be an open cover of $\bigcup_{i=1}^{n} K_i$, so $\bigcup_{i=1}^{n} K_i \subset \bigcup \theta$.
Clearly θ is an open cover of each K_j. Since each K_j is compact, it has finite subcover $\{\theta_{j,1}, \theta_{j,2}, \dots, \theta_{j,m_j}\}$.
But that means

$$\{\theta_{1,1}, \theta_{1,2}, \dots, \theta_{1,m_1}, \theta_{2,1}, \theta_{2,2}, \dots, \theta_{2,m_2}, \dots, \theta_{n,1}, \theta_{n,2}, \dots, \theta_{n,m_n}\}$$

is a finite subcover of θ that covers $\bigcup_{i=1}^{n} K_i$, completing the proof. ∎

Ex 4.4.4 Show that if K compact and $C \subset K$ is closed, then C is compact.

Proof: Let θ be an open cover of C, then let $\tilde{\theta}$ be θ with $\mathbb{R}\setminus C$ included.
By Proposition 4.3.5, we see that $\tilde{\theta}$ is an open cover of K.
Since K compact, $\tilde{\theta}$ has a finite subcover $\{\theta_1, \theta_2, \dots, \theta_m\}$, i.e.
$C \subset K \subset \bigcup_{i=1}^{m} \theta_i$, so we see that C is compact. ∎



Ex 4.4.5 If for all $\alpha \in A$, K_α is compact, so that $\bigcap_{\alpha \in A} K_\alpha$ is compact.

Proof: Let θ be an open cover of $\bigcap_{\alpha\in A} K_\alpha$. Let $\gamma \in A$. Then extend θ to $\tilde{\theta}$ by including $\mathbb{R}\setminus K_\gamma$. Then $\tilde{\theta}$ is an open cover of the compact set K_γ, so $\tilde{\theta}$ has a finite subcover $\{\theta_{\gamma_1}, \theta_{\gamma_2}, \dots, \theta_{\gamma_m}\}$. But then

$$\bigcap_{\alpha\in A} K_\alpha \subset K_\gamma \subset \bigcup_{i=1}^{m} \theta_{\gamma_i},$$

hence $\{\theta_{\gamma_1}, \dots, \theta_{\gamma_m}\}$ is a finite subcover of $\bigcap_{\alpha\in A} K_\alpha$, completing the proof. ∎

Ex 4.4.6 Show $K \subset \mathbb{R}$ compact iff every infinite subset of K has a limit pt in K.

Proof: Suppose $K \subseteq \mathbb{R}$ compact and let $J \subset K$ be an infinite subset of K. Pick an increasing sequence $\{x_1, x_2, \dots\} \subset K$.
But K is closed and bounded. By Thm 2.1.1, the sequence $\{x_i\}$ converges to some limit K, and since K is closed Prop 4.3.2 shows $x \in K$ and x is a limit point of K.

Conversely, suppose every infinite subset $J \subset K$ has a limit point in K. Let $\{x_n\}$ be a sequence in K.
Choose $J = \{x_1, x_2, \dots\}$ which is an infinite subset of K, hence it has a limit point $x \in K$.
Thus there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x \in K$ and so Prop 4.4.7 shows K is compact. ∎