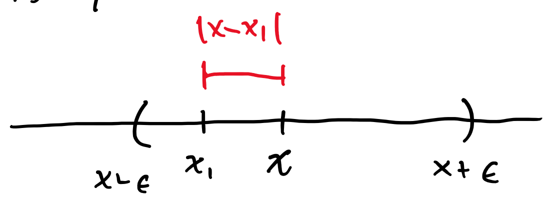
HW10 MTH 427/527 Fall 2024

Tuesday, November 5, 2024 11:48 AM

Ex 4.3.2 Suppose x is a limit point of A. Show $\forall \epsilon > 0$, the set $(x-\epsilon, x+\epsilon) \cap A$ is infinite.

Proof: Since x is a limit point, we know $\forall \epsilon > 0$ $(x-\epsilon, x+\epsilon) \cap A$ is nonempty.
Choose $x_1 \in (x-\epsilon, x+\epsilon) \cap A$. Choose $\epsilon_1 = \frac{|x_1 - x|}{2}$, i.e. half the distance between x and x_1.
Since x is a limit point, $(x-\epsilon_1, x+\epsilon_1) \cap A$ is nonempty and $x_1 \notin (x-\epsilon_1, x+\epsilon_1)$ by choice of ϵ_1.
So pick $x_2 \in (x-\epsilon_1, x+\epsilon_1) \cap A$.



In general, pick some $x_n \in (x-\epsilon_{n-1}, x+\epsilon_{n-1}) \cap A$ and choose $\epsilon_n = \frac{|x-x_n|}{2}$.
Doing this generates an infinite sequence $x_1, x_2, x_3, \dots \in A$. ∎

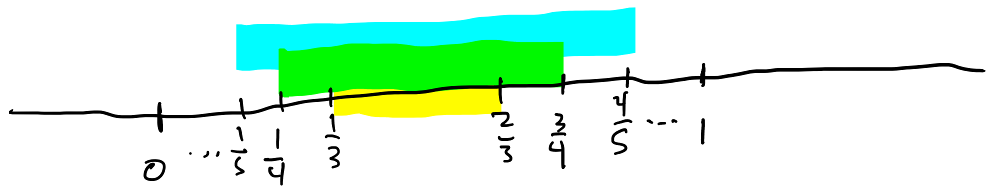
Ex 4.3.5 For $n = 3, 4, 5, \dots$ $I_n = \left[\frac{1}{n}, \frac{n-1}{n}\right]$. Is $\bigcup_{n=3}^{\infty} I_n$ open or closed?

$I_3 = \left[\frac{1}{3}, \frac{2}{3}\right]$ $\quad \frac{1}{n} \to 0$

$I_4 = \left[\frac{1}{4}, \frac{3}{4}\right]$ $\quad \frac{n-1}{n} \to 1$

$I_5 = \left[\frac{1}{5}, \frac{4}{5}\right]$

⋮

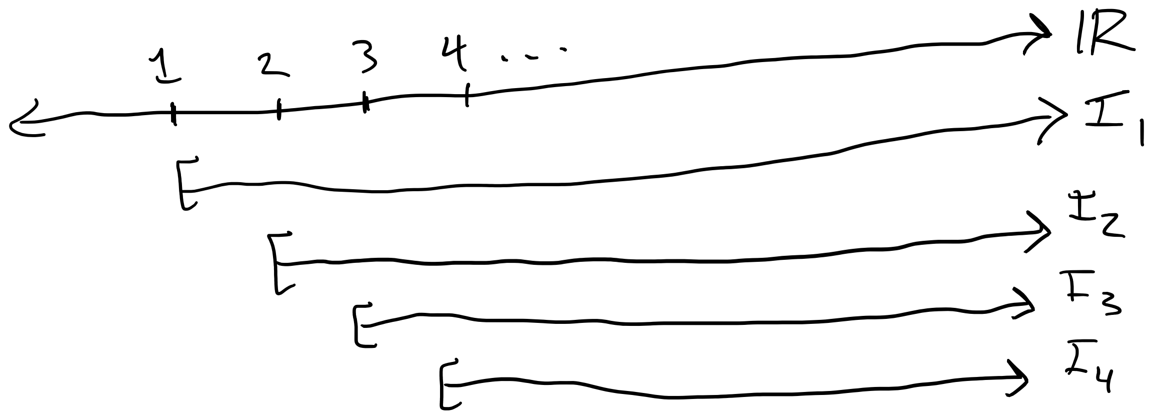


Notice that the intervals grow and tend to $[0,1]$.
But is $0 \in \bigcup_{n=3}^{\infty} I_n$ and $1 \in \bigcup_{n=3}^{\infty} I_n$?
No! No one of the I_n sets contain 0 or 1.

Thus,
$$\bigcup_{n=3}^{\infty} I_n = (0,1) \text{ which is open}$$

Ex 4.3.7 Find sequence I_n, $n = 1, 2, 3, \dots$ of closed intervals such that $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Soln: Let $I_n = [n, \infty)$ be a sequence of closed intervals. Then,
$$\bigcap_{n=1}^{\infty} I_n = \emptyset$$



We know the I_n are closed because they contain all their limit points!
It is clear that $I_{n+1} \subset I_n$!

Ex 4.3.8 Find sequence of bounded open intervals such that $I_{n+1} \subset I_n$ and $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

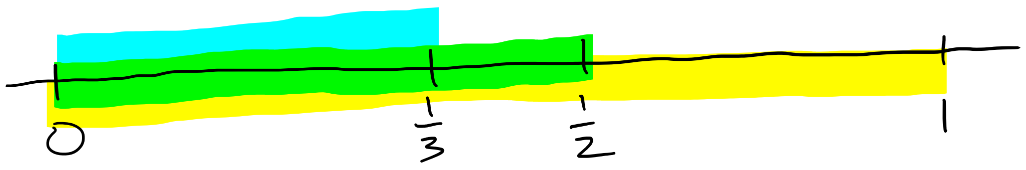
Soln: Let $I_n = \left(0, \frac{1}{n}\right)$ for $n = 1, 2, 3, \dots$
Then $I_{n+1} \subseteq I_n$ and each I_n is open and bounded.

$I_1 = (0,1)$

$I_2 = \left(0, \frac{1}{2}\right)$

$I_3 = \left(0, \frac{1}{3}\right)$

⋮



No $\tau > 0$ remains in the intersection, since for any such τ, $\exists n$ s.t. $\frac{1}{n} < \tau$.