Quiz 7 MTH 335

Wednesday, November 13, 2024 2:52 PM

Solve $y'' + 2y' + 2y = \delta(t-\pi)$, $y(0) = 1$, $y'(0) = 0$

Soln: Take $\mathcal{L}$ of both sides (using table!) to get

$$\left(s^2 Y(s) - s(1) - 0\right) + 2\left(sY(s) - 1\right) + 2Y(s) = e^{-\pi s}$$

$$\underbrace{(s^2 + 2s + 2)}Y(s) = e^{-\pi s} + s + 2$$

does not nicely factor, so complete the square:

$$s^2 + 2s + 2 + \left(\tfrac{2}{2}\right)^2 - \left(\tfrac{2}{2}\right)^2$$
$$= (s+1)^2 + (2-1)$$
$$= (s+1)^2 + 1$$

Completing the square

$$x^2 + \alpha x + \beta$$
$$= x^2 + \alpha x + \beta + \left(\tfrac{\alpha}{2}\right)^2 - \left(\tfrac{\alpha}{2}\right)^2$$
$$= \underbrace{\left(x + \tfrac{\alpha}{2}\right)^2 + \left(\beta - \left(\tfrac{\alpha}{2}\right)^2\right)}$$

check

$$\left(x + \tfrac{\alpha}{2}\right)^2 = x^2 + 2\left(\tfrac{\alpha}{2}\right)x + \left(\tfrac{\alpha}{2}\right)^2$$
$$+ \left(\beta - \left(\tfrac{\alpha}{2}\right)^2\right)$$
$$= x^2 + \alpha x + \beta \ ✓✓$$

Therefore,

move one to the other fraction

$$Y(s) = \frac{e^{-\pi s}}{(s+1)^2 + 1} + \frac{s}{(s+1)^2 + 1} + \frac{2}{(s+1)^2 + 1}$$

$$= \frac{e^{-\pi s}}{(s+1)^2 + 1} + \frac{s+1}{(s+1)^2 + 1} + \frac{1}{(s+1)^2 + 1}$$

How to invert?

First notice $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t)$ and $\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos(t)$

By 1st translation thm $\mathcal{L}\{F(s-a)\} = e^{at} f(t)$ we have

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-(-1))^2+1}\right\} = e^{-t}\sin(t) \quad a = -1$$

and

$$\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{s-(-1)}{(s-(-1))^2+1}\right\} = e^{-t}\cos(t)$$

By 2nd translation thm $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,\mathcal{U}(t-a)$ we have

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{(s+1)^2+1}\right\} = e^{-(t-\pi)}\cos(t-\pi)\,\mathcal{U}(t-\pi)$$

Therefore,

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{(s+1)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= e^{-(t-\pi)}\cos(t-\pi)\,\mathcal{U}(t-\pi) + e^{-t}\cos(t) + e^{-t}\sin(t)$$