Quiz 6 MTH 335 Fall 2024

Wednesday, October 2, 2024 10:46 AM

Compute $\mathcal{L}\{t\}(s)$ using integration by parts.

$$\mathcal{L}\{t\}(s) \overset{\text{def}}{=} \int_0^\infty t e^{-st}\,dt = \lim_{b\to\infty} \int_0^b t e^{-st}\,dt$$

$\int u\,dv = uv - \int v\,du$

To compute the integral, we use integration by parts:

$$\int_0^b t e^{-st}\,dt = -\frac{t}{s} e^{-sb}\Big|_{t=0}^{t=b} - \int_0^b \left(-\frac{1}{s}\right) e^{-st}\,dt$$

choose: $u = t \quad dv = e^{-st}$

compute: $du = dt \quad v = -\frac{1}{s} e^{-st}$

$$= \left(-\frac{b}{s} e^{-sb} + 0\right) + \frac{1}{s} \int_0^b e^{-st}\,dt$$

$w = -st$

$dw = -s\,dt \rightarrow dt = -\frac{1}{s}\,dw$

$$= \int_0^{-\infty} \left(-\frac{1}{s}\right) e^u\,du = -\frac{1}{s} \int_0^{-\infty} e^u\,du = -\frac{1}{s} \lim_{b\to\infty} \int_0^{-b} e^u\,du$$

$$= -\frac{1}{s}\left[\lim_{b\to\infty} e^u \Big|_0^{-b}\right]$$

$$= -\frac{1}{s} \lim_{b\to\infty} \left[e^{-b} - 1\right]$$

$$= -\frac{1}{s}[0 - 1] = \frac{1}{s}$$

$$= \frac{-b}{s} e^{-sb} + \frac{1}{s}\left(\frac{1}{s}\right)$$

$$= -\frac{b}{s} e^{-sb} + \frac{1}{s^2}$$

Therefore, we have

$$\mathcal{L}\{t\}(s) = \lim_{b\to\infty} \int_0^b t e^{-st}\,dt$$

$$= \lim_{b\to\infty} \left[\cancel{-\frac{b}{s} e^{-sb}} + \frac{1}{s^2}\right]$$

0 by L'Hôpital's rule!

$$= \frac{1}{s^2}$$