Quiz 5 MTH 335 Fall 2024

Wednesday, October 2, 2024 10:30 AM

Given that $y_1(t) = \frac{1}{t}$ solves $2t^2 y'' + ty' - 3y = 0$, find the other independent soln. using reduction of order.

Soln: Assume $y_2(t) = u(t)y_1(t) = \frac{u(t)}{t}$, where $u(t)$ is an unknown function.

Compute $y_2'(t) = \frac{tu'(t) - u(t)}{t^2}$ and $y_2''(t) = \frac{t^2[\cancel{u'(t)} + tu''(t) - \cancel{u'(t)}] - [tu'(t) - u(t)]2t}{t^4}$

$$= \frac{t^3u''(t) - 2t^2u'(t) + 2tu(t)}{t^4}$$

Now substitute y_2, y_2', and y_2'' into the differential equation to obtain

$$2t^2\left[\frac{t^3u''(t) - 2t^2u'(t) + 2tu(t)}{t^4}\right] + t\left[\frac{tu'(t) - u(t)}{t^2}\right] - 3\frac{u(t)}{t} = 0$$

$$\left[2\frac{t^3}{t^2}u''(t) - 4u'(t) + \cancel{4\frac{u(t)}{t}}\right] + \left[u'(t) - \cancel{\frac{u(t)}{t}}\right] - 3\cancel{\frac{u(t)}{t}} = 0$$

$$2tu''(t) - 3u'(t) = 0$$

Let $w = u'$ so $w' = u''$. Then this becomes

$2tw'(t) - 3w(t) = 0 \longleftarrow$ separable 1^{st} order ODE

$$2t\frac{dw}{dt} = 3w$$

$$\int \frac{1}{w}\,dw = \int \frac{3}{2t}\,dt$$

$$\ln(w) = \frac{3}{2}\ln(t) + C = \ln(t^{3/2}) + C$$

$$\downarrow$$

$$w = e^{\ln(t^{3/2}) + C} = e^C e^{\ln(t^{3/2})} = \tilde{C}t^{3/2}, \quad \tilde{C} = e^C$$

Therefore, the general solution is

$$y(t) = \frac{c_1}{t} + c_2 t^{3/2}$$