HW9 MTH 300 Fall 2024

Monday, November 11, 2024 9:12 AM

Ch. 5 #6 Suppose $x \in \mathbb{R}$. If $x^3 - x > 0$, then $x > -1$.

Proof: Suppose $x \leq -1$. Then $x^2 \geq 0 \geq -x$, so $-x^2 \leq x$. Now multiply $x^2 \geq -x$ by x to get

$$x^3 \leq -x^2 \leq x$$

by contrapositive

This shows $x^3 - x \leq 0$, completing the proof. ∎

#18 If $a, b \in \mathbb{Z}$, then $(a+b)^3 \equiv a^3 + b^3 \bmod 3$

$a \equiv b \bmod 3$ iff $3 \mid a-b$

Proof: First calculate

$$(a+b)^3 = a^3 + 3ba^2 + 3b^2 a + b^3$$

Thus we see that

$$(a+b)^3 \equiv a^3 + b^3 \bmod 3$$

because

$$(a+b)^3 - (a^3+b^3) = (a^3 + 3ba^2 + 3b^2a + b^3) - (a^3+b^3) = 3(ba^2 + b^2 a)$$

is divisible by 3. ∎

Ch. 6 #9 Suppose $a, b \in \mathbb{R}$. If a is rational and ab is irrational, then b is irrational.

Proof: Suppose that b is rational, so $\exists p, q \in \mathbb{Z}$ so that $b = \frac{p}{q}$. But a is rational, so $\exists k, \ell \in \mathbb{Z}$ so that $a = \frac{k}{\ell}$. But then

$$ab = \left(\frac{k}{\ell}\right)\left(\frac{p}{q}\right) = \frac{kp}{\ell q}$$

but since $kp \in \mathbb{Z}$ and $\ell q \in \mathbb{Z}$, this means ab is rational, a contradiction.

Therefore b is irrational. ∎