HW8 MTH 300 Fall 2024

Sunday, October 13, 2024 3:05 PM

Ch.4: #15, 16; Ch.5: #1, 3, 4, 5

Ch.4

16. If two integers have the same parity, then their sum is even. (Try cases.)

Proof: Let $x, y \in \mathbb{Z}$.

Case 1: Suppose both x and y are even. Then $\exists m, n \in \mathbb{Z}$ so that $x = 2m$ and $y = 2n$. Then,

$$x + y = 2m + 2n = 2(m+n)$$

and since $m, n \in \mathbb{Z}$, $m+n \in \mathbb{Z}$, so we see that $x+y$ is even, completing the proof in this case.

Case 2: Suppose both x and y are odd. Then $\exists m, n \in \mathbb{Z}$ so that $x = 2m+1$ and $y = 2n+1$. Then

$$x + y = (2n+1) + (2m+1) = 2n + 2m + 2 = 2(m+n+1)$$

and since $m, n \in \mathbb{Z}$, $m+n+1 \in \mathbb{Z}$, so we see that $x+y$ is even, completing the proof in this case.

Since both cases exhausted all possibilities, the proof is complete. ∎

Ch.5

"not even" is odd

1. Suppose $n \in \mathbb{Z}$. If n^2 is even, then n is even.

Proof (by contrapositive): Suppose n is odd. Then $\exists m \in \mathbb{Z}$ so that $n = 2m+1$.

Then $n^2 = (2m+1)^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$

Since $m \in \mathbb{Z}$, $2m^2 + 2m \in \mathbb{Z}$, so we see that n^2 is odd, completing the proof. ∎

3. Suppose $a, b \in \mathbb{Z}$. If $a^2(b^2 - 2b)$ is odd, then a and b are odd.

Proof (by contrapositive): Suppose it's not the case that a and b are odd, i.e. that a is even or b is even.

$\sim(A \wedge B) = \sim A \vee \sim B$

Case 1: Suppose a is even. Then $\exists l \in \mathbb{Z}$ so that $a = 2l+1$. Then

$$\begin{aligned} a^2(b^2 - 2b) &= (2l+1)^2(b^2 - 2b) \\ &= (4l^2 + 4l + 1)(b^2 - 2b) \\ &= 4l^2b^2 - 8l^2b + 4lb^2 - 8lb - 2b \\ &= 2(2l^2b^2 - 4l^2b + 2lb^2 - 4lb - b) \end{aligned}$$

Since $l, b \in \mathbb{Z}$, we have $2l^2b^2 - 4l^2b + 2lb^2 - 4lb - b \in \mathbb{Z}$, so we see that $a^2(b^2 - 2b)$ is even, completing the proof in this case.

Case 2: Suppose b is even, then $\exists l \in \mathbb{Z}$ so that $b = 2l$. Then

$$\begin{aligned} a^2(b^2 - 2b) &= a^2((2l)^2 - 2(2l)) \\ &= a^2(4l^2 - 4l) \\ &= 2[a^2(2l^2 - 2l)] \end{aligned}$$

Since $a, l \in \mathbb{Z}$, we see that $a^2(2l^2 - 2l) \in \mathbb{Z}$, so we have that $a^2(b^2 - 2b)$ is even, completing the proof in this case.

Since all possibilities have been considered, the proof is complete. ∎

4. Suppose $a, b, c \in \mathbb{Z}$. If a does not divide bc, then a does not divide b.

Proof (by contrapositive). Let $a, b, c \in \mathbb{Z}$ and suppose a divides b, i.e. that $\exists l \in \mathbb{Z}$ so that $b = la$. Then,

$$bc = (la)c = a(lc)$$

Since $l, c \in \mathbb{Z}$, $lc \in \mathbb{Z}$, and so we see that a divides bc, completing the proof. ∎

5. Suppose $x \in \mathbb{R}$. If $x^2 + 5x < 0$ then $x < 0$.

Proof (by contrapositive): Let $x \in \mathbb{R}$. Assume that $x \geq 0$. Then, $x^2 \geq 0$ and $5x \geq 0$, so we conclude that $x^2 + 5x \geq 0$, completing the proof. ∎