HW7 MTH300 Fall 2024

Sunday, October 13, 2024 2:48 PM

Chapter 4: #1, 3, 4, 5, 8, 9, 14

1. If x is an even integer, then x^2 is even.

Proof: Let x be even, so there is some integer a so that $x=2a$.

Then

$$x^2=(2a)^2=4a^2=2(2a^2)$$

But since $a\in\mathbb{Z}$, $2a^2\in\mathbb{Z}$, so we see that x^2 is even, completing the proof. ∎

3. If a is an odd integer, then a^2+3a+5 is odd.

Proof: Let $a\in\mathbb{Z}$ be odd, so there exists $l\in\mathbb{Z}$ such that $a=2l+1$.

Now compute

$$\begin{aligned}a^2+3a+5 &= (2l+1)^2+3(2l+1)+5\\ &=(4l^2+4l+1)+(6l+3)+5\\ &=4l^2+10l+9\\ &=4l^2+10l+8+1\\ &=2(2l^2+5l+4)+1\end{aligned}$$

Since $l\in\mathbb{Z}$, $2l^2+5l+4\in\mathbb{Z}$, so we see that a^2+3a+5 is odd, completing the proof. ∎

4. Suppose $x,y\in\mathbb{Z}$. If x and y are odd, then xy is odd.

Proof: Let $x,y\in\mathbb{Z}$ be odd, so there are $m,n\in\mathbb{Z}$ such that $x=2m+1$ and $y=2n+1$.

Then,

$$\begin{aligned}xy=(2m+1)(2n+1) &= 4mn+2m+2n+1\\ &=2(2mn+m+n)+1\end{aligned}$$

Since $m,n\in\mathbb{Z}$, $2mn+m+n\in\mathbb{Z}$, so we see that xy is odd, completing the proof. ∎

5. Suppose $x,y\in\mathbb{Z}$. If x is even, then xy is even.

Proof: Let $x,y\in\mathbb{Z}$ with x even. Then there is some $m\in\mathbb{Z}$ so that $x=2m$.

Then

$xy=(2m)y=2(my)$ and since $m,y\in\mathbb{Z}$, $my\in\mathbb{Z}$, so we see that xy is odd, completing the proof. ∎

14. If $n\in\mathbb{Z}$, then $5n^2+3n+7$ is odd. (Try cases.)

Proof: Let $n\in\mathbb{Z}$.

Case 1: Suppose n is even, i.e. that $\exists m\in\mathbb{Z}$ so that $n=2m$. Then,

$$\begin{aligned}5n^2+3n+7 &= 5(2m)^2+3(2m)+7\\ &=5(4m^2)+6m+7\\ &=20m^2+6m+6+1\\ &=2(10m^2+3m+3)+1\end{aligned}$$

Since $m\in\mathbb{Z}$, $10m^2+3m+3\in\mathbb{Z}$, so we see that $5n^2+3n+7$ is odd, completing the proof in this case.

Case 2: Suppose n is odd, so $\exists l\in\mathbb{Z}$ so that $n=2l+1$. Then,

$$\begin{aligned}5n^2+3n+7 &=5(2l+1)^2+3(2l+1)+7\\ &=5[4l^2+4l+1]+(6l+3)+7\\ &=20l^2+20l+5+6l+3+7\\ &=20l^2+26l+15\\ &=20l^2+26l+14+1\\ &=2(10l^2+13l+7)+1\end{aligned}$$

Since $l\in\mathbb{Z}$, $10l^2+13l+7\in\mathbb{Z}$, so we see that $5n^2+3n+7$ is odd, completing the proof in this case.

Since we obtained the conclusion and both cases exhaust all options, the proof is complete. ∎