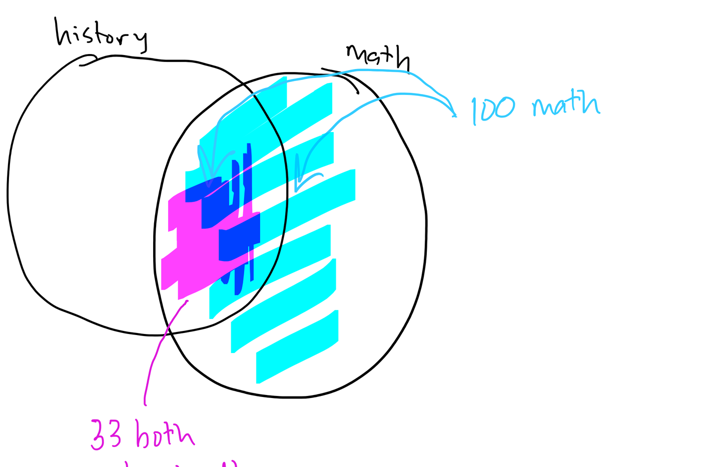
HW6 MTH300 Fall 2024

Sunday, October 13, 2024 1:06 PM

Section 3.7: #1, 6, 10, 11; Section 3.8: #1, 2, 5, 16, 18; Section 3.9: #1, 2, 6; Section 3.10: #2, 3, 4, 5

§3.7

1. At a certain university 523 of the seniors are history majors or math majors (or both). There are 100 senior math majors, and 33 seniors are majoring in both history and math. How many seniors are majoring in history?



total

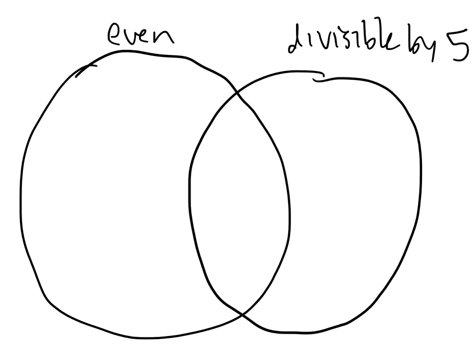
$523 = |\text{history}| + |\text{math}| - |\text{history} \cap \text{math}|$

$= |\text{history}| + 100 - 33 \longrightarrow |\text{history}| = 523 - 100 + 33$

$= 423 - 33$

$= 396$ history majors

10. How many 6-digit numbers are even or are divisible by 5?



_ _ _ _ _ _ $\Rightarrow 9 \cdot 10^5 = 900000$ total numbers

9 choices (no 0), 10 choices

even $\longrightarrow$ $\underline{9}\ \underline{10}\ \underline{10}\ \underline{10}\ \underline{10}\ \underline{5} = (45) \cdot 10^4 = 450000$ even #'s

can be 0,2,4,6,8

divisible by 5 $\underline{9}\ \underline{10}\ \underline{10}\ \underline{10}\ \underline{10}\ \underline{2} = 18 \cdot 10^4 = 180000$ divisible by 5

can be 5,0

even + div by 5 $\underline{9}\ \underline{10}\ \underline{10}\ \underline{10}\ \underline{10}\ \underline{1} = 9 \cdot 10^4 = 90000$

must be 0

$|\text{even} \cup \text{div by } 5| = |\text{even}| + |\text{div by } 5| - |\text{even} \cap \text{div by } 5|$

$= 450000 + 180000 - 90000 = 540000$

$\binom{k+n-1}{k}$

§3.8

1. How many 10-element multisets can be made from the symbols $\{1,2,3,4\}$?

By Fact 3.7: $n=4, k=10$

$\Downarrow$

$\binom{10+4-1}{10} = \binom{13}{10} = \frac{13!}{10!\,3!} = \frac{13 \cdot 12 \cdot 11}{3 \cdot 2 \cdot 1}$

2. How many 2-element multisets can be made from the 26 letters of the alphabet?

By Fact 3.7: $n = 26, k = 2$

$\downarrow$

$\binom{26+2-1}{2} = \binom{27}{2} = \frac{27!}{2!\,25!} = \frac{27 \cdot 26}{2}$

16. A community in Canada's Northwest Territories is known in the local language as "TUKTUYAAQTUUQ." How many permutations does this name have?

3 T's, 4 U's, 2 A's, 2 Q's

Total of 13 letters

$\xrightarrow{\text{Fact 3.8}} \frac{13!}{3!\,4!\,2!\,2!}$ ways

§3.9

1. Show that if six numbers are chosen at random, then at least two of them will have the same remainder when divided by 5.

2. You deal a pile of cards, face down, from a standard 52-card deck. What is the least number of cards the pile must have before you can be assured that it contains at least five cards of the same suit?

#1 remainder when divided by 5 can be 0, 1, 2, 3, or 4

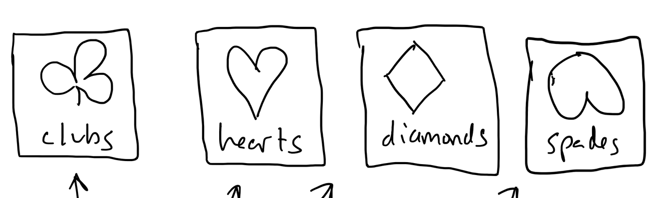
so you have 6 numbers (pigeons), 5 options for those numbers (holes)

$\downarrow$

$\boxed{R=0}$ $\boxed{R=1}$ $\boxed{R=2}$ $\boxed{R=3}$ $\boxed{R=4}$

more pigeons than holes $\longrightarrow$ at least one hole has two pigeons

#2 4 suits $\longrightarrow$



possible: 4 in each $\longleftarrow$ 20 cards

But next choice guarantees one has one of those suits

$\Downarrow$

21 cards guarantees it

§3.10

3. Show that $\binom{n}{2}\binom{n-2}{k-2} = \binom{n}{k}\binom{k}{2}$.

4. Show that $P(n,k) = P(n-1,k) + k \cdot P(n-1,k-1)$.

5. Show that $\binom{2n}{2} = 2\binom{n}{2} + n^2$.

#3

$\binom{n}{2}\binom{n-2}{k-2} = \left(\frac{n!}{2!\,\cancel{(n-2)!}}\right)\left(\frac{\cancel{(n-2)!}}{(k-2)!\,\underbrace{((n-2)-(k-2))!}_{(n-k)!}}\right) = \frac{n!}{2!\,(k-2)!\,(n-k)!}$

$\binom{n}{k}\binom{k}{2} = \left(\frac{n!}{\cancel{k!}\,(n-k)!}\right)\left(\frac{\cancel{k!}}{2!\,(k-2)!}\right) = \frac{n!}{(n-k)!\,2!\,(k-2)!}$

equal!

#4 fact 3.4: $P(n,k) = \frac{n!}{(n-k)!}$

they match!

$P(n-1,k) + k\,P(n-1,k-1) = \frac{(n-1)!}{(n-1-k)!} + k\,\frac{(n-1)!}{\underbrace{((n-1)-(k-1))!}_{(n-k)! = (n-k)(n-k-1)!}} = \frac{(n-1)!}{(n-k-1)!}\Big[\underbrace{1 + \frac{k}{n-k}}_{\frac{n-k+k}{n-k} = \frac{n}{n-k}}\Big] = \frac{n!}{(n-k-1)!\,(n-k)} = \frac{n!}{(n-k)!}$

#5

$\binom{2n}{2} = \frac{(2n)!}{2!\,(2n-2)!} = \frac{(2n)(2n-1)}{2} = \frac{4n^2 - 2n}{2} = 2n^2 - n$

match!

$2\binom{n}{2} + n^2 = \cancel{2}\,\frac{n!}{\cancel{2!}(n-2)!} + n^2 = n(n-1) + n^2 = n^2 - n + n^2 = 2n^2 - n$