HW5 MTH 300 Fall 2024

Sunday, October 13, 2024 12:03 PM

Section 3.4: #7, 8, 12, 13; Section 3.5: #1, 2, 3, 8, 11, 12, 13; Section 3.6: #3, 4, 5, 9, 12

§3.4 #7 #7-digit #'s made from 1,2,3,4,5,6,7,8,9

no repetition

all odd digits to left of all even

_ _ _ _ _ _ _

could have 3 or 5 odd digits (b/c only 4 evens)

$\underset{\text{odd}}{5\cdot 4\cdot 3}\,\Big|\,\underset{\text{even}}{4\cdot 3\cdot 2\cdot 1}$ $\quad \underset{\text{odd}}{5\cdot 4\cdot 3}\,\Big|\,\underset{\text{even}}{4\cdot 3\cdot 2}$ $\quad \underset{\text{odd}}{5\cdot 4\cdot 3\cdot 2\cdot 1}\,\Big|\,\underset{\text{even}}{4\cdot 3}$

$= 60\cdot 24 + 120\cdot 24 + 120\cdot 12 = 1440 + 2880 + 240 = 4560$ ways

$$\begin{array}{r} 13 \\ 4\overline{)52} \\ \underline{4} \\ 12 \\ \underline{-12} \\ 39 \end{array}$$

#12 7 cards drawn from 52-card deck, # lists w/ no club?

There are 13 clubs $\longrightarrow$ $52-13=39$ non-club cards

$\underset{\substack{\uparrow \\ 39 \\ \text{options}}}{_}\ \underset{\substack{\uparrow \\ 38}}{_}\ \underset{\substack{\uparrow \\ 37}}{_}\ \underset{\substack{\uparrow \\ 36}}{_}\ \underset{\substack{\uparrow \\ 35}}{_}\ \underset{\substack{\uparrow \\ 34}}{_}\ \underset{\substack{\uparrow \\ 33}}{_} = 39\cdot 38\cdot 37\cdot 36\cdot 35\cdot 34\cdot 33 = P(39,7) = \dfrac{39!}{32!}$

$= 77519922480$ ways

ok to drop if are of time on exam

#13 lists length 6 from 26 letters?

$\underset{\substack{\uparrow \\ 26}}{_}\ \underset{25}{_}\ \underset{24}{_}\ \underset{23}{_}\ \underset{22}{_}\ \underset{21}{_} = P(26,6) = \dfrac{26!}{20!}$ ways

§3.5

1. Suppose a set A has 37 elements. How many subsets of A have 10 elements? How many subsets have 30 elements? How many have 0 elements?

10 elts: $C(37,10) = \dfrac{37!}{10!\,27!}$

30 elts: $C(37,30) = \dfrac{37!}{30!\,7!}$

0 elts: $C(37,0) = \dfrac{37!}{0!\,37!} = 1$ way

3. A set X has exactly 56 subsets with 3 elements. What is the cardinality of X?

Call cardinality of X n: i.e. let n be so that $|X| = n$

We are told

$C(n,3) = 56$

$\dfrac{n!}{3!\,(n-3)!} = 56$

$\dfrac{n(n-1)(n-2)}{\underset{=6}{3!}} = 56 \longrightarrow \dfrac{n\overbrace{(n-1)(n-2)}^{n^2-3n+2}}{\text{expand}} = 56\cdot 6 = 336$

$n^3 - 3n^2 + 2n = 336$

solutions are

$n = 8,\ -\frac{5}{2} \pm \frac{\sqrt{143}}{2}\,i$

any polynomial you get on exam will be linear or quadratic!

$$\begin{array}{r} 56 \\ \underline{6} \\ 336 \end{array}$$

1st digit not 0

11. How many positive 10-digit integers contain no 0's and exactly three 6's?

12. Twenty-one people are to be divided into two teams, the Red Team and the Blue Team. There will be 10 people on Red Team and 11 people on Blue Team. In how many ways can this be done?

13. Suppose $n, k \in \mathbb{Z}$, and $0 \le k \le n$. Use Fact 3.5, the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, to show that $\binom{n}{k} = \binom{n}{n-k}$.

#11 ways to pick exactly 3 locations for the 6's: $C(10,3) = \dfrac{10!}{3!\,7!} = \dfrac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1} = \dfrac{720}{6} = 120$

7 other spots: Can have $\underbrace{1,2,3,4,5,7,8,9}_{8 \text{ choices}}$ in them

8^7 ways to pick remaining #'s

$\Rightarrow (120)(8^7)$ ways

$$\begin{array}{r} 120 \\ 6\overline{)720} \\ \underline{6} \\ 120 \\ \underline{-12} \\ 00 \end{array}$$

#12 pick red team: $C(21,10)$ + pick blue team from remaining 11: $C(11,11) = \dfrac{21!}{10!\,11!} + \underbrace{\dfrac{11!}{11!\,0!}}_{=1}$ ways

#13 Suppose $0 \le k \le n$ and recall $\binom{a}{b} = \dfrac{a!}{b!\,(a-b)!}$

$a=n$, $b=k$ $\longrightarrow \binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$

equal!

$a=n$, $b=n-k$ $\longrightarrow \binom{n}{n-k} = \dfrac{n!}{(n-k)!\,(n-(n-k))!} = \dfrac{n!}{(n-k)!\,k!}$

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

3. Use the binomial theorem to find the coefficient of x^8 in $(x+2)^{13}$.

4. Use the binomial theorem to find the coefficient of x^6y^3 in $(3x-2y)^9$.

5. Use the binomial theorem to show $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

#3 $(x+2)^{13} = \sum_{k=0}^{13} \binom{13}{k} x^k 2^{13-k}$

we care about the $k=8$ term: $\binom{13}{8} 2^{13-8} x^8 = \boxed{\binom{13}{8} 2^5} x^8$

ok to leave unsimplified

$= \dfrac{13!}{8!\,5!} \cdot 2^5 = \left(\dfrac{13\cdot 12\cdot 11\cdot 10\cdot 9}{5\cdot 4\cdot 3\cdot 2\cdot 1}\right)(32)$ ways

#4 $(3x-2y)^9 = \sum_{k=0}^{9} \binom{9}{k} (3x)^k (-2y)^{9-k}$

we care about the $k=6$ term

$\longrightarrow \binom{9}{6} (3x)^6 (-2y)^{9-6} = \underbrace{\left(\dfrac{9!}{6!\,3!}\right) 3^6 (-2)^3}_{\text{the requested number}} x^6 y^3$

#5 $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k}$

key insight! binomial thm simplify

9. Use the binomial theorem to show $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \binom{n}{4} - \cdots + (-1)^n \binom{n}{n} = 0$, for $n > 0$.

$0 = 0^n = (-1+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n}$

true key insight binomial thm

12. Show that $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$.

$\binom{n}{k}\binom{k}{m} = \left(\dfrac{n!}{\cancel{k!}\,(n-k)!}\right)\left(\dfrac{\cancel{k!}}{m!\,(k-m)!}\right) = \left(\dfrac{n!}{(n-k)!}\right)\left(\dfrac{1}{m!\,(k-m)!}\right)$

match!

$\binom{n}{m}\binom{n-m}{k-m} = \left(\dfrac{n!}{m!\,\cancel{(n-m)!}}\right)\left(\dfrac{\cancel{(n-m)!}}{(k-m)!\,\underbrace{((n-m)-(k-m))!}_{(n-k)!}}\right) = \left(\dfrac{n!}{m!}\right)\left(\dfrac{1}{(k-m)!\,(n-k)!}\right)$