HW13 MTH 300 Fall 2024

Section 11.1 #2: Consider the relation $R = \{(a,b),(a,c),(c,c),(b,b),(c,b),(b,c)\}$ on the set $A = \{a,b,c\}$. Is R reflexive? Is R symmetric? Is R transitive?
Solution: No, R is not reflexive, because $(a,a) \notin R$. No, R is not symmetric because $(a,c) \in R$ but $(c,a) \notin R$. Yes R is transitive.

Section 11.1 #12 Prove that the relation $|$ (divides) on the set $\mathbb{Z}$ is reflexive and transitive.
Solution: To show that it is reflexive, we note that for any $x \in \mathbb{Z}$, $x|x$ because choosing $k = 1 \in \mathbb{Z}$ yields $x = kx$. To show $|$ is transitive, let $a|b$ (hence there is $k \in \mathbb{Z}$ so that $b = ak$) and $b|c$ (hence there is $\ell \in \mathbb{Z}$ so that $c = b\ell$). Then

$$c = b\ell = (ak)\ell = a(k\ell),$$

showing that $a|c$. Thus $|$ is transitive.
Section 11.1 #14 Suppose that R is a symmetric transitive relation on a set A, and there is some $a \in A$ so that aRx for all $x \in A$. Prove that R is reflexive.
Solution: Let $x \in R$. We know that aRx and since R is transitive, we know xRa. So we have xRa and aRx and by the transitive property we obtain xRx. Since x was an arbitrary element of A, we have shown that R is reflexive. ■

Section 11.2 #2 Let $A = \{a,b,c,d,e\}$. Suppose R is an equivalence relation on A. Suppose R has two equivalence classes. Also aRd, bRc, and eRd. write out R as a set.
Solution: Since eRd and R symmetric, we have dRa. Since we have eRd and dRa, we have eRa. This shows that $\{e,a,d\} \subset [a]$. Since bRc, we know that $\{b,c\} \subset [c]$. Since equivalence classes are disjoint (meaning their intersection is empty or the whole set) and we know that there are two equivalence classes, and $A = \{e,a,d\} \cup \{b,c\}$, we have shown that $[a] = \{e,a,d\}$ and $[c] = \{b,c\}$.

Section 11.2 #10: Suppose R and S are two equivalence relations on a set A. Prove that $R \cap S$ is also an equivalence relation.
Solution: Since both R and S are equivalence relations, for every $a \in A$, $(a,a) \in R$ and $(a,a) \in S$, hence $(a,a) \in R \cap S$; thus $R \cap S$ is reflexive. Since for all $(a,b) \in R$, $(b,a) \in R$ and for all $(c,d) \in S$, $(d,c) \in S$, for any element $(e,f) \in R \cap S$ $(f,e) \in R \cap S$. Finally, if $a,b,c \in A$ with $(a,b),(b,c) \in R \cap S$, since (a,b) and (b,c) are in both the equivalence relations R and S, we know that (a,c) is in both R and S, hence $(a,c) \in R \cap S$, completing the proof. ■

Section 11.3 #4 Let $A = \{a,b,c,d,e\}$. Suppose R is an equivalence relation on A. Suppose also that aRd, bRc, eRa, and cRe. How many equivalence classes does R have?
Solution: Since we have eRa and aRd, we obtain eRd by transitivity (so $\{e,a,d\} \subset [a]$). Since we have bRc and cRe, we have bRe by transitivity (so $\{b,c,e\} \subset [b]$. But this mean $[a] \cap [b]$ is nonempty, so in fact $[a] = [b]$

by a theorem. Since all elements of A were in either $[a]$ or $[b]$, we have that $[a] = [b] = \{a, b, c, d, e\}$, i.e. there is exactly one equivalence class.

Section 11.3 #8 Define a relation R on $\mathbb{Z}$ as xRy if and only if x^2+y^2 is even. Prove R is an equivalence relation.

Proof: First we show that R is reflexive. Given any $x \in R$, $x^2 + x^2 = 2x^2$ is even, thus $(x, x) \in R$ for all $x \in \mathbb{Z}$, hence R is reflexive. If $(x, y) \in R$, it means that x^2+y^2 is even, but since $x^2+y^2 = y^2+x^2$ then also must be even, we have $(y, x) \in R$, showing that R is symmetric. Finally, if $(x, y) \in R$ and $(y, z) \in R$, it means that $x^2 + y^2$ is even and $y^2 + z^2$ is even. So there exist $k, \ell \in \mathbb{Z}$ so that $x^2 + y^2 = 2k$ and $y^2 + z^2 = 2\ell$. Therefore, we see that $x^2 = 2k - y^2$ and $z^2 = 2\ell - y^2$. Thus, we compute

$$x^2 + z^2 = (2k - y^2) + (2\ell - y^2) = 2k + 2\ell - 2y^2 = 2(k + \ell - y^2),$$

showing that $x^2 + z^2$ is even. ■

Section 11.5 #4 Write out the addition and multiplication tables for $\mathbb{Z}_6$.

Solution:

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

·	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1