HW12 MTH 300 Fall 2024

Chapter 10 #2: Prove that

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proof: The formula is equivalent to

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

First check the base case $n = 1$: the left-hand side

$$\sum_{k=1}^{1} k^2 = 1^2 = 1,$$

while the right-hand side equals

$$\frac{1(1+1)(2\cdot 1+1)}{6} = \frac{1\cdot 2\cdot 3}{6} = \frac{6}{6} = 1.$$

Since the left-hand side and the right-hand side are equal, the base case is proven.

Now we make our inductive hypothesis: assume that for some $n = N$,

$$(*) \qquad \sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}.$$

What we need to show is that

$$\sum_{k=1}^{N+1} k^2 = \frac{(N+1)((N+1)+1)(2(N+1)+1)}{6} = \frac{(N+1)(N+2)(2N+3)}{6}.$$

So compute

$$\begin{aligned}
\sum_{k=1}^{N+1} k^2 &= \left(\sum_{k=1}^{N} k^2\right) + (N+1)^2 \\
&\overset{(*)}{=} \frac{N(N+1)(2N+1)}{6} + (N+1)^2 \\
&= (N+1)\left[\frac{N(2N+1)}{6} + N + 1\right] \\
&= (N+1)\left[\frac{2N^2 + N + 6N + 6}{6}\right] \\
&= (N+1)\left[\frac{2N^2 + 7N + 6}{6}\right]
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left[\frac{(N+1)(N+2)(2N+3)}{6}\right] &= (N+1)\left[\frac{(N+2)(2N+3)}{6}\right] \\
&= (N+1)\left[\frac{2N^2 + 3N + 4N + 6}{6}\right] \\
&= (N+1)\left[\frac{2N^2 + 7N + 6}{6}\right],
\end{aligned}$$

and so we are done. ■

Chapter 10 #5: If $n \in \mathbb{N}$, then

$$2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2.$$

Proof: The claim can be written in summation notation as

$$\sum_{k=1}^{n} 2^k = 2^{n+1} - 2.$$

Find the base case $n = 1$ is true:

$$\sum_{k=1}^{n} 2^k = 2^1 \stackrel{?}{=} 2^{1+1} - 2$$

$$2 \stackrel{\checkmark}{=} 4 - 2.$$

Now assume as inductive hypothesis that

$$(*) \qquad \sum_{k=1}^{N} 2^k = 2^{N+1} - 2.$$

We need to show that

$$\sum_{k=1}^{N+1} 2^k = 2^{(N+1)+1} - 2 = 2^{N+2} - 2.$$

So compute

$$\begin{aligned} \sum_{k=1}^{N} 2^{N+1} &= \left(\sum_{k=1}^{N} 2^k\right) + 2^{N+1} \\ &\stackrel{(*)}{=} \left(2^{N+1} - 2\right) + 2^{N+1} \\ &= 2 \cdot 2^{N+1} - 2 \\ &= 2^{N+2} - 2, \end{aligned}$$

completing the proof. ■

Chapter 10 #11: Prove that $3|(n^3 + 5n + 6)$ for every integer $n \geq 0$.

Proof: First prove the base case $n = 0$: it is true because $3|(0^3 + 5(0) + 6)$, i.e. $3|6$. Now assume the inductive hypothesis $3|(N^3 + 5N + 6)$, and we need to show that $3|\big((N+1)^3 + 5(N+1) + 6\big)$. By the inductive hypothesis, we know there exists $k \in \mathbb{Z}$ so that

$$(*) \qquad 3 = (N^3 + 5N + 6).$$

Now compute

$$\begin{aligned} (N+1)^3 + 5(N+1) + 6 &= \big(N^3 + 3N^2 + 3N + 1\big) + 5N + 5 + 6 \\ &= \big(N^3 + 5N + 6\big) + 3N^2 + 5 + 1 \\ &= 3k + 3N^2 + 6 \\ &= 3\big(k + N^2 + 2\big), \end{aligned}$$

which means that $3|\big((N+1)^3+5(N+1)+6\big)$, completing the proof. ■

Chapter 10 #26 For the Fibonacci sequence F_k, where $F_1 = F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$, show that

$$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}.$$

Proof: The base case becomes $F_1^2 = F_1 F_2$, which is true because $F_1 = F_2 = 1$. Now assume for inductive hypothesis that

$$(*) \qquad \sum_{k=1}^{N} F_k^2 = F_N F_{N+1}.$$

We need to show that

$$\sum_{k=1}^{N+1} F_k^2 = F_{N+1} F_{N+2}.$$

So, compute

$$\begin{aligned} \sum_{k=1}^{N+1} F_k^2 &= \left(\sum_{k=1}^{N} F_k^2\right) + F_{N+1}^2 \\ &\overset{(*)}{=} F_{N+1}F_{N+2} + F_{N+1}^2 \\ &= F_{N+1}\big(F_N + F_{N+1}\big) \end{aligned}$$

Recall the Fibonaci sequence's defining formula (for all n, $F_{n+1} = F_n + F_{n-1}$) says that for $n = N+1$, $F_{N+2} = F_{N+1} + F_N$, so we have shown

$$\sum_{k=1}^{N+1} F_k^2 = F_{N+1}\big(F_N + F_{N+1}\big) = F_{N+1}F_{N+2},$$

completing the proof. ■