HW11 MTH 300 Fall 2024

Monday, November 11, 2024 10:13 AM

Ch. 8 #7 Let A, B, C be sets. If $B \subseteq C$, then $A \times B \subseteq A \times C$.

Proof: Let $(a,b) \in A \times B$. Since $B \subseteq C$, $b \in C$ and thus $(a,b) \in A \times C$, completing the proof. ■

#8 Show $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof: ($\subseteq$) Let $x \in A \cup (B \cap C)$, so $x \in A$ or $x \in B \cap C$.

If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$

If $x \in B \cap C$, then $x \in B$ and $x \in C$, so $x \in A \cup B$ and $x \in A \cup C$.

Thus in either case,

$$A \cup (B \cup C) \subseteq (A \cup B) \cap (A \cup C).$$

($\supseteq$) Let $x \in (A \cup B) \cap (A \cup C)$. Then $x \in A \cup B$ and $x \in A \cup C$.

If $x \in A$, then $x \in A \cup (B \cap C)$.

If $x \notin A$, then $x \in B$ and $x \in C$, so $x \in B \cap C$, hence $x \in A \cup (B \cap C)$.

Thus in either case,

$$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C).$$

So we showed

$$A \cup (B \cup C) \subseteq (A \cup B) \cap (A \cup C) \text{ and } (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C),$$

hence we have shown $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. ■

#11 Let A, B be sets in a universal set U.

$\overline{A \cup B} = \bar{A} \cap \bar{B}$

$\bar{A} = U - A$

$\bar{B} = U - B$

Proof: ($\subseteq$) Let $x \in \overline{A \cup B}$, so $x \in U - (A \cup B)$. This means $x \in U - A = \bar{A}$ and $x \in U - B = \bar{B}$, hence $x \in \bar{A} \cap \bar{B}$, showing $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$.

($\supseteq$) Let $x \in \bar{A} \cap \bar{B}$, so $x \in \bar{A} = U - A$ and $x \in \bar{B} = U - B$, so $x \notin A$ and $x \notin B$.

Thus $x \in U - (A \cup B) = \overline{A \cup B}$, showing $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$.

By both parts, we have shown $\overline{A \cup B} = \bar{A} \cap \bar{B}$. ■

#12 Show $A - (B \cap C) = (A - B) \cup (A - C)$.

Proof: ($\subseteq$) Let $x \in A - (B \cap C)$, so $x \in A$ and $x \notin B \cap C$, i.e. $x \notin B$ and $x \notin C$.

Hence $x \in A - B$ and $x \in A - C$. Thus

$$A - (B \cap C) \subseteq (A - B) \cup (A - C).$$

($\supseteq$) Let $x \in (A - B) \cup (A - C)$, so either $x \in A - B$ or $x \in A - C$.

WLOG assume $x \in A - B$, i.e. that $x \in A$ and $x \notin B$.

Thus $x \notin B \cap C$, so $x \in A - (B \cap C)$. Thus,

$$(A - B) \cup (A - C) \subseteq A - (B \cap C).$$

By both parts, we showed

$$A - (B \cap C) = (A - B) \cup (A - C). \quad ■$$

Ch. 9 #3 "if $n \in \mathbb{Z}$ and $n^5 - n$ is even, then n is even"

is false because for $n = 3$, $n^5 - n = 3^5 - 3 = 243 - 3 = 240$ is even

odd

#7 "if $A \times C = B \times C$, then $A = B$"

is true

Proof: (by contradiction) ($\rightarrow$) Suppose $A \neq B$ and $A \times C = B \times C$.

WLOG assume $\exists x \in A$ such that $x \notin B$.

Then any $(x, c) \in A \times C$ cannot also be in $A \times B$, a contradiction. Thus we have shown $A = B$.

($\leftarrow$) Suppose $A = B$. Then

$$A \times C = \{(a,c) : a \in A, c \in C\}$$
$$= \{(b,c) : b \in B, c \in C\} = B \times C,$$

completing the proof. ■

#11 "if $a, b \in \mathbb{N}$, then $a + b < ab$" is false:

Let $a = b = 2 \in \mathbb{N}$, then $a + b = 4$ and $ab = 4$ so we see $a + b < ab$ does not hold.

#13 "there exists X so that $\mathbb{R} \subseteq X$ and $\emptyset \subseteq X$" is true

Simply take $X = \mathbb{R}$

#14 "$\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$" is true

Proof:

$X \in \mathcal{P}(A) \cap \mathcal{P}(B)$

iff

$X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$

iff

$X \subseteq A$ and $X \subseteq B$

iff

$X \subseteq A \cap B$

iff

$X \in \mathcal{P}(A \cap B)$

structure shows how we would prove

$\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$

and

$\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$