HW10 MTH 300 Fall 2024

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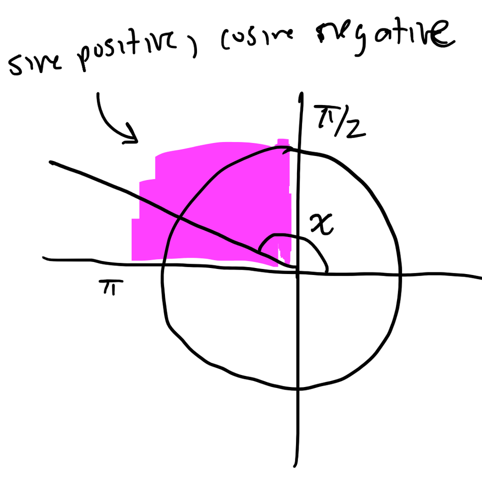

Ch. 6 #13 For every $x \in [\frac{\pi}{2}, \pi]$, $\sin(x) - \cos(x) \geq 1$.

Proof: Suppose $\sin(x) - \cos(x) < 1$.

We know

$$\left.\begin{array}{l}\sin(x) \geq 0 \\ \cos(x) \leq 0 \implies -\cos(x) \geq 0\end{array}\right\} \implies \sin(x)\cos(x) \leq 0$$

Thus

$$\sin(x) - \cos(x) \geq 0.$$

Fact
if
$0 < y < 1$
then $0 < y^2 < 1$

Therefore,

$0 \leq \sin(x) - \cos(x) < 1$, so we conclude that

$$0 \leq (\sin(x) - \cos(x))^2 < 1$$

$$0 \leq \underbrace{\sin^2(x) - 2\sin(x)\cos(x) + \cos^2(x)}_{=1} < 1, \text{ so}$$

$$0 \leq 1 - 2\sin(x)\cos(x) < 1, \text{ so}$$

$$-2\sin(x)\cos(x) < 0, \text{ so divide by } -2 \text{ to get}$$

$$\sin(x)\cos(x) > 0,$$

but this is a contradiction, completing the proof. ∎

#14 If A and B are sets, then $A \cap (B - A) = \emptyset$.

Proof: If $x \in A \cap (B-A)$, it means $x \in A$ and $x \in B - A$, but this cannot happen because $B - A$ contains no points of A. Thus no such x can exist, so $A \cap (B - A)$ is empty. ∎

#24 $\log_2(3)$ is irrational

Proof: Suppose $\log_2(3)$ is rational, i.e. $\exists p, q \in \mathbb{Z}^+$ so that (both can be ⊕ b/c $\log_2(3) > 0$)

$$\log_2(3) = \frac{p}{q}$$

Then

$$\cancel{2}^{\cancel{\log_2}(3)} = 2^{\frac{p}{q}}, \text{ so}$$

$$3 = 2^{\frac{p}{q}}$$

Taking power q, we get $3^q = 2^p$, but this cannot be the case because the left-hand side is not divisible by 2 while the right-hand side is divisible by 2, completing the proof. ∎

Chapter 7 #2 Suppose $x \in \mathbb{Z}$. Then x odd iff $3x + 6$ is odd.

Proof: ($\rightarrow$) Suppose x is odd, i.e. $x = 2k + 1$ for some $k \in \mathbb{Z}$.

Then,
$$3x + 6 = 3(2k+1) + 6 = 6k + 3 = 6k + 2 + 1$$
$$= 2(3k+1) + 1,$$

so we see that $3x + 6$ is odd.

($\leftarrow$) Conversely, we will prove the contrapositive of "if $3x+6$ is odd, then x is odd", i.e. we will prove "if x is even, then $3x+6$ is even".

So assume x is even, i.e. $x = 2k$ for some $k \in \mathbb{Z}$.

Then compute

$$3x + 6 = 3(2k) + 6 = 6k + 6 = 2(3k+3),$$

showing $3x + 6$ is even, completing the proof in this direction. ∎

#9 Suppose $a \in \mathbb{Z}$. Prove $14 \mid a$ iff $7 \mid a$ and $2 \mid a$.

Proof: ($\rightarrow$) Suppose $14 \mid a$, i.e. $a = 14k$ for some $k \in \mathbb{Z}$.

But then $a = 7(2k)$, hence $7 \mid a$

and $a = 7(2k)$, hence $2 \mid a$,

completing the proof in this direction.

($\longleftarrow$) Suppose $7 \mid a$ and $2 \mid a$, so $\exists k, l \in \mathbb{Z}$ such that $a = 7k$ and $a = 2l$.

Since $2 \mid 7k$ and $2 \nmid 7$, we know that $2 \mid k$ so $\exists m \in \mathbb{Z}$ so that $k = 2m$.

Therefore,

$$a = 7k = 7(2m) = 14m$$

and so $14 \mid a$, completing the proof in this direction. ∎

Ch. 8 #1 Prove $\{12n : n \in \mathbb{Z}\} \subseteq \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$

Proof: Let $x \in \{12n : n \in \mathbb{Z}\}$ so $x = 12k$ for some $k \in \mathbb{Z}$.

But then

$$x = 12k = 2(6k) \text{ and } x = 12k = 3(4k).$$

Since $6k \in \mathbb{Z}$, we see that $x \in \{2n : n \in \mathbb{Z}\}$.

Since $4k \in \mathbb{Z}$, we see that $x \in \{3n : n \in \mathbb{Z}\}$.

Therefore $x \in \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$, completing the proof. ∎

#2 Prove $\{6n : n \in \mathbb{Z}\} = \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$

Proof: ($\subseteq$) Let $x \in \{6n : n \in \mathbb{Z}\}$, so $x = 6k$ for some $k \in \mathbb{Z}$.

Then

$$x = 6k = 2(3k) \text{ and } x = 6k = 3(2k)$$

so $x \in \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$. Therefore $\{6n : n \in \mathbb{Z}\} \subseteq \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$.

($\supseteq$) Let $x \in \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}$. So we know x is even and $x = 3k$ for some $k \in \mathbb{Z}$. But $2 \mid x = 3k$ and $2 \nmid 3$, so $2 \mid k$, i.e. $k = 2l$ for some $l \in \mathbb{Z}$.

Thus,

$$x = 3k = 3(2l) = 6l,$$

hence $x \in \{6n : n \in \mathbb{Z}\}$. Therefore, $\{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\} \subseteq \{6n : n \in \mathbb{Z}\}$.

Since we showed

$$\{6n : n \in \mathbb{Z}\} \subseteq \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\} \text{ and}$$

$$\{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\} \subseteq \{6n : n \in \mathbb{Z}\},$$

hence

$$\{6n : n \in \mathbb{Z}\} = \{2n : n \in \mathbb{Z}\} \cap \{3n : n \in \mathbb{Z}\}. \quad ∎$$