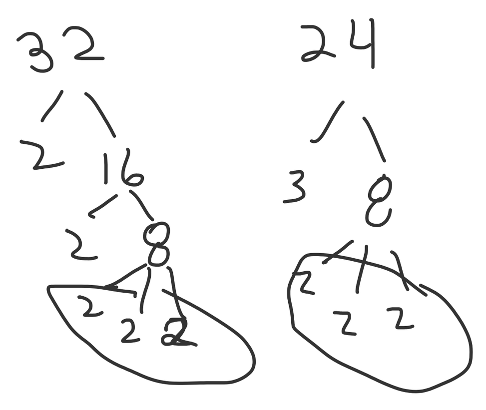
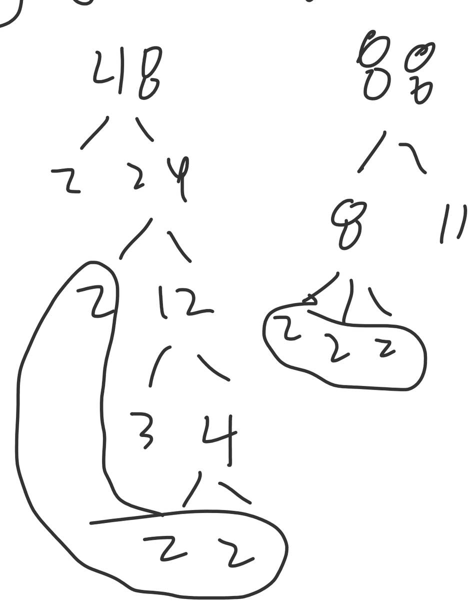
HW7 MTH450.550 Fall 2023

Sunday, November 12, 2023 1:45 PM

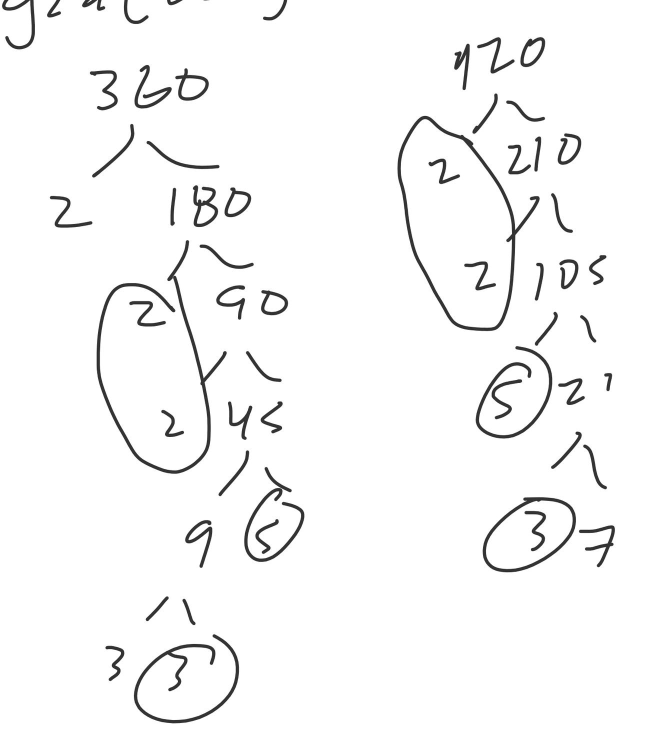
p.66 #5) $\gcd(32,24)=8$



#6) $\gcd(48,88)=8$



#7) $\gcd(360,420)=2^2\cdot 3\cdot 5=60$



#17) Subgroup of $\mathbb{Z}_{30}$ generated by 25

$25^2 = \cancel{25}+25 \bmod 30 = 50 \bmod 30 = 20$

$25^3 = 20+25 \bmod 30 = 45 \bmod 30 = 15$

$25^4 = 15+25 \bmod 30 = 40 \bmod 30 = 10$

$25^5 = 10+25 \bmod 30 = 35 \bmod 30 = 5$

$25^6 = 5+25 \bmod 30 = 30 \bmod 30 = 0$

$\Rightarrow \langle 25\rangle = \{25,20,15,10,5,0\}$

#18) Subgroup of $\mathbb{Z}_{42}$ gen by 30

$30^2 = 60 \bmod 42 = 18$

$30^3 = 48 \bmod 42 = 6$

$30^4 = 36$

$30^5 = 66 \bmod 42 = 24$

$30^6 = 54 \bmod 42 = 12$

$30^7 = 42 \bmod 42 = 0$

$\Rightarrow \langle 30\rangle = \{30,18,6,36,24,12,0\}$

#25) $\mathbb{Z}_6$ has subgroups of sizes 1, 2, 3, and 6 — divisors of 6

$\langle 0\rangle$ $\langle 3\rangle$ $\langle 2\rangle$ $\langle 1\rangle$

#26) $\mathbb{Z}_8$ has subgroups of sizes 1, 2, 4, 8

$\langle 0\rangle$ $\langle 4\rangle$ $\langle 2\rangle$ $\langle 1\rangle$

#44) Let G be cyclic with generator a, and let G' be isomorphic to G. If $\phi: G\to G'$ is an isomorphism, show that for every $x\in G$, $\phi(x)$ is determined by $\phi(a)$.

Proof: Let $x\in G$ and assume that $\phi(a)$ is known.

Since G cyclic, $\exists\, m \in \mathbb{Z}$ so that $a^m = x$.

Thus

$$\phi(x) = \phi(a^m) = \phi(\overbrace{a*a*a*\cdots*a}^{m \text{ times}})$$

$$= \underbrace{\phi(a)*'\phi(a)*'\phi(a)*'\cdots*'\phi(a)}_{m \text{ times}}$$

structure preserving property multiple times

$$= \phi(a)^m$$

So the value of $\phi(x)$ is just a power of $\phi(a)$. In other words, the value of $\phi(a)$ completely determines the value of $\phi(x)$. ∎

Grad students:

#46) Let $a,b\in G$. If ab has finite order n, then so does ba.

Proof: Suppose $|ab|=n$, i.e. that $(ab)^n=e$ and no power $<n$ yields e, where e is the group identity.

Since $(ab)^n=e$,

$$e = \underbrace{(ab)(ab)(ab)\cdots(ab)(ab)}_{n \text{ times}}$$

$$= a\underbrace{(ba)(ba)(ba)\cdots(ba)(ba)}_{n-1 \text{ times}}b$$

$$= a(ba)^{n-1}b$$

Therefore,

$$a(ba)^{n-1}b = e$$

mult left a^{-1}, right b^{-1}

$$\boxed{(ba)^{n-1} = a^{-1}b^{-1}} \quad (\star)$$

Now need to show $(ba)^n = e$.

But,

$$(ba)^n = (ba)(ba)^{n-1}$$

$$\overset{\star}{=} (ba)(a^{-1}b^{-1}) = b(aa^{-1})b^{-1} = beb^{-1} = bb^{-1} = e$$