HW3 MTH 450/550 Fall 2023

p.34 #4

one-to-one: $\phi(m)=\phi(n) \leftrightarrow m+1=n+1$

$\leftrightarrow m=n$

Thus ϕ is 1-1

onto: choose $m \in \mathbb{Z}$

Does $\exists n \in \mathbb{Z}$ s.t. $\phi(n) \overset{?}{=} m$?

$\updownarrow$

$n+1 \overset{?}{=} m$

$\updownarrow$

$n = m-1 \in \mathbb{Z}$ ✓

Thus ϕ is onto

structure-preserving:

Does $\phi(m+n) \overset{?}{=} \phi(m)+\phi(n)$?

$\updownarrow$

$m+n+1 \overset{?}{=} (m+1)+(n+1)$

$\updownarrow ?$

$m+n+1 \overset{?}{=} m+n+2$

$\updownarrow ?$

$1 \overset{?}{=} 2$ – false

So ϕ is not order-preserving.

Thus ϕ is not an isomorphism.

p.34 #5

1-1: $\phi(a)=\phi(b) \leftrightarrow \frac{a}{2}=\frac{b}{2} \leftrightarrow a=b \rightarrow$ ✓

onto: Let $b \in \mathbb{Q}$. Does $\exists\ a \in \mathbb{Q}$ such that $\phi(a) \overset{?}{=} b$?

$\updownarrow$

$\frac{a}{2}=b \leftrightarrow a=2b \in \mathbb{Q}$ ✓

str. pres.: $\phi(a+b) \overset{?}{=} \phi(a)+\phi(b)$

$\updownarrow ?$

$\frac{a+b}{2} \overset{?}{=} \frac{a}{2}+\frac{b}{2}$

yes, true! ✓

Thus ϕ is an isomorphism.

p.34 #6

one-to-one:

$\phi(a)=\phi(b) \leftrightarrow a^2=b^2$

$\leftrightarrow a = \pm\sqrt{b}$

not a unique answer... suggests not one-to-one

See: $\phi(2)=\phi(-2)$

$\downarrow$

$4=4$

Thus ϕ not 1-1

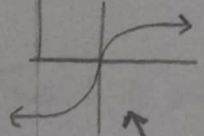
Thus ϕ not isomorphism.

p.34 #7

1-1: $\phi(a)=\phi(b) \leftrightarrow a^3=b^3$

$\leftrightarrow a=b$ ✓ yes 1-1

recall that graph of $y=\sqrt[3]{x}$ looks like



it's one-to-one! because it is monotone increasing!

onto: Let $a \in \mathbb{R}$.

Does $\exists b \in \mathbb{R}$ s.t.

$\phi(b) \overset{?}{=} a$

$\updownarrow$

$b^3 \overset{?}{=} a$

$\updownarrow$

$\boxed{b=\sqrt[3]{a}} \in \mathbb{R}$

Str. pres:

$\phi(a \cdot b) \overset{?}{=} \phi(a)\cdot\phi(b)$

$(a \cdot b)^3 \overset{?}{=} a^3 \cdot b^3$

$\uparrow$

yes! true! ✓

Thus ϕ is an isomorphism.

?

p.34 #11 one-to-one: $\phi(f)=\phi(g)$

$$\Updownarrow$$
$$f'=g'$$
$$\Updownarrow \int_a^x$$
$$\int_a^x f'(t)\,dt=\int_a^x g'(t)\,dt$$
$$\Updownarrow$$
$$f(x)-f(a)=g(x)-g(a)$$
$$\Updownarrow$$
$$f(x)=\underbrace{g(x)-g(a)+f(a)}$$

unless $g(a)=-f(a)$, we do not arrive at $f=g$

Thus ϕ not one-to-one

Thus ϕ not isomorphism.

p.34 #12 1-1: $\phi(f)=\phi(g)$

$$\Updownarrow$$
$$f'(0)=g'(0)$$

But there are many functions that are different while obeying this condition.

Think:

If $f(x)=x^2$ and $g(x)=x^3$, then

$f'(x)=2x$ and $g'(x)=3x^2$, so

$f'(0)=2(0)=0=3(0^2)=g'(0)$

So ϕ is not an isomorphism, since it is not 1-1.

p.34 #13 1-1: $\phi(f)=\phi(g)$ means

for all $x\in\mathbb{R}$, $(\phi(f))(x)=(\phi(g))(x)$

$$\Updownarrow$$

for all $x\in\mathbb{R}$, $\int_0^x f(t)\,dt=\int_0^x g(t)\,dt$

$$\frac{d}{dx}\Downarrow$$

for all $x\in\mathbb{R}$, $f(x)=g(x)$ (by the fundamental theorem of calculus)

Thus $f=g$ and so ϕ is 1-1.

(#13 cont)

onto: Let $g\in F$, then we know $g:\mathbb{R}\to\mathbb{R}$ and $g^{(n)}$ exists for all $n\in\{0,1,2,3,\dots\}$

Does $\exists f\in F$ s.t. $\phi(f)=g$?

$$\Updownarrow$$

for all $x\in\mathbb{R}$, $\int_0^x f(t)\,dt=g(x)$

$$\Downarrow \frac{d}{dt}$$
$$f(x)=g'(x)$$

Does it work?

Compute for $f=g'$,

$$\phi(f)=\int_0^x f(t)\,dt$$
$$=\int_0^x g'(t)\,dt$$
$$=g(t)\Big|_0^x=\underbrace{g(x)-g(0)}$$

unless $g(0)=0$ is required (it isn this ain't just g

Thus f is not onto.

(notice: def of ϕ tells us thi immediately! for any $h\in F$,

$$(\phi(h))(0)=\int_0^0 h(t)\,dt=0$$

p.34 #14

one-to-one: $\phi(f) = \phi(g)$

$\Updownarrow$

$\forall x \in \mathbb{R}, \ (\phi(f))(x) = (\phi(g))(x)$

$\Updownarrow$

$\forall x \in \mathbb{R}, \ \frac{d}{dx}\int_0^x f(t)\,dt = \frac{d}{dx}\int_0^x g(t)\,dt$

$\Updownarrow$ (by FTC)

$\forall x \in \mathbb{R}, \ f(x) = g(x)$

$\Updownarrow$

$f = g$

onto: Let $g \in F$. Then we know $g: \mathbb{R} \to \mathbb{R}$ and $g^{(n)}$ exists for all $n = 0, 1, 2, \ldots$

Does $\exists f \in F$ s.t.

$\phi(f) = g$?

$\Updownarrow$?

for all $x \in \mathbb{R}$, $\frac{d}{dx}\int_0^x f(t)\,dt = g(x)$

$\Updownarrow$

$f(x) = g(x)$ ✓

So you see that ϕ appears to just be the "identity map" from F to F because of FTC. This map could be considered the "trivial isomorphism on F."

order-preserving $\phi(f+g) \overset{?}{=} \phi(f) + \phi(g)$

$\Updownarrow$

$\forall x \in \mathbb{R} \quad \frac{d}{dx}\int_0^x f(t)+g(t)\,dt \overset{?}{=} \underbrace{\frac{d}{dx}\int_0^x f(t)\,dt}_{=f(x)} + \underbrace{\frac{d}{dx}\int_0^x g(t)\,dt}_{=g(x)}$

$\Updownarrow$

$f(x)+g(x) \overset{\checkmark}{=} f(x) + g(x)$ true!

So ϕ is an isomorphism.

p.34 #8

Not an isomorphism.

ϕ is not 1-1, e.g.

$\phi\left(\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\right) = \phi\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right)$

while

$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

p.34 #9

Yes it is an isomorphism.

If $M \in M_1(\mathbb{R})$, then

$M = [x]$, where $[x]$ denotes a 1×1 matrix,

So, $\det(M) = \det([x]) = x$

It is easy to see that this map is 1-1 and onto.

That it preserves order is just observing for

$M, N \in M_1(\mathbb{R})$ where $M = [x]$ and $N = [y]$

$\phi(M \cdot N) = \det(M \cdot N)$

$= \det([x]\cdot[y])$

$= \det([xy])$

$= xy$

$= \phi(M)\,\phi(N)$

p.34 #10

1-1: $\phi(x)=\phi(y)$

$\Updownarrow$

$0.5^x = 0.5^y$

$\Updownarrow$

$\ln(0.5^x) = \ln(0.5^y)$

$\Updownarrow$

$x\cancel{\ln(0.5)} = y\cancel{\ln(0.5)}$

$\Updownarrow$

$x=y$ ✓

onto: Let $y \in \mathbb{R}^+$.

Does $\exists x \in \mathbb{R}$ so that $\phi(x)=y$?

$\Updownarrow$

$0.5^x = y$

$x = \frac{\ln(y)}{0.5} = 2\underbrace{\ln(y)}$ ✓

(exists because $y>0$)

strpres: $\phi(a+b) = 0.5^{a+b}$

$= (0.5^a)\cdot(0.5^b)$

$= \phi(a)\cdot\phi(b)$

Thus ϕ is an isomorphism. ✓

p.34 #15

1-1: $\phi(f) = \phi(g)$

$\forall x \in \mathbb{R},\ xf(x) = xg(x)$

$x\neq 0$: $f(x)=g(x)$ ✓

$x=0$: $0f(0) = 0g(0)$ true whether or not $f(0)=g(0)$

$\Downarrow$

unless $f(0)=g(0)$, $f\neq g$

$\Downarrow$

ϕ not 1-1