Written HW4 – MATH 3503 Fall 2022

The so-called "Fresnel integrals" are special functions given by the formulas $S(t) = \int_0^t \sin\left(x^2\right) \mathrm{d}x$ and $C(t) = \int_0^t \cos\left(x^2\right) \mathrm{d}x$. They have applications to optics and are related to the "normal distribution" from probability theory. Indeed, it is not possible to find an elementary antiderivative of these functions, so the integral is our best representation of them.

If we consider the parametric curve

$$\begin{cases} \vec{r}(t) = \langle C(t), S(t) \rangle \\ -\infty < t < \infty \end{cases},$$

then its plot is called the "Cornu spiral" (or "Euler spiral").

In this homework, we will draw the Cornu spiral as well as approximate it with a power series to see some limitations in that approximation.

1. First plot the Cornu spiral in Desmos. (*hint*: recall that parametric curves can be plotted using $(f(t), g(t))$ in Desmos. If you type "int" (without the quotes) then an integral will appear for you to use).

2. Let us devise a way to approximate this curve with a polynomial. Recall that $k! = k(k-1)(k-2)\dots(3)(2)(1)$ Recall from calculus 2 the Taylor series for sine and cosine:

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

   and

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

   Take the first five terms of this series and write that polynomial for this question. Call the first five terms of the $\sin(x)$ "$G(x)$" and call the first five terms of $\cos(x)$ "$H(x)$".

3. Compute $G(x^2)$ and $H(x^2)$.

4. Calculate $L(t) = \int_0^t G(x^2)\mathrm{d}x$ and $M(t) = \int_0^t H(x^2)\mathrm{d}x$.

5. Use Desmos to plot $\langle M(t), L(t) \rangle$ for $-10 \le t \le 10$ and use the picture as part of the response to this question. What do you notice about this plot versus the plot from earlier in this homework description? What would you have to do to improve the approximation?

6. Finally, go back to the original definitions of $C(t)$ and $S(t)$ and compute the curvature κ of $\vec{r}(t) = \langle C(t), S(t) \rangle$ in any way you choose.