Euler method

Fact: $x'(t) = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$

If we fix h to be small nonzero ($h=0.001$), then this becomes

$$\frac{x(t+h) - x(t)}{h}$$

So $\begin{cases} x'(t) = \sin(\tan(x)) \\ x(0) = 1 \end{cases}$

Euler method: pick $h = \overset{\text{"small"}}{0.001}$ + rewrite

$$\frac{x(t+h) - x(t)}{h} = \sin(\tan(x(t)))$$

$$\boxed{x(t+h) = h\sin(\tan(x(t))) + x(t)}$$

$t=0$: $x(0) = 1$

$t=0 \rightarrow x(0+h) = h\sin(\tan(x(0))) + x(0)$

$= h\sin(\tan(1)) + 1 \approx \textcircled{\#} \leftarrow$

$\boxed{t=h} \rightarrow x(0+2h) = h\sin(\tan(\underset{\#}{\underline{x(h)}})) + \underset{\#}{\underline{x(h)}}$

$\vdots$

Sequence of values

$x(0), x(h), x(2h), \ldots$

Lotka-Volterra (predator-prey models)

$(r, a, m, b > 0)$

$$\begin{cases} x' = \underbrace{rx - axy}_{f(x,y)} \\ y' = \underbrace{-my + bxy}_{g(x,y)} \end{cases}$$

x ~ prey pop

y - pred pop

x-null $x' = 0 \rightarrow 0 = rx - axy$

$= (x)(r - ay)$

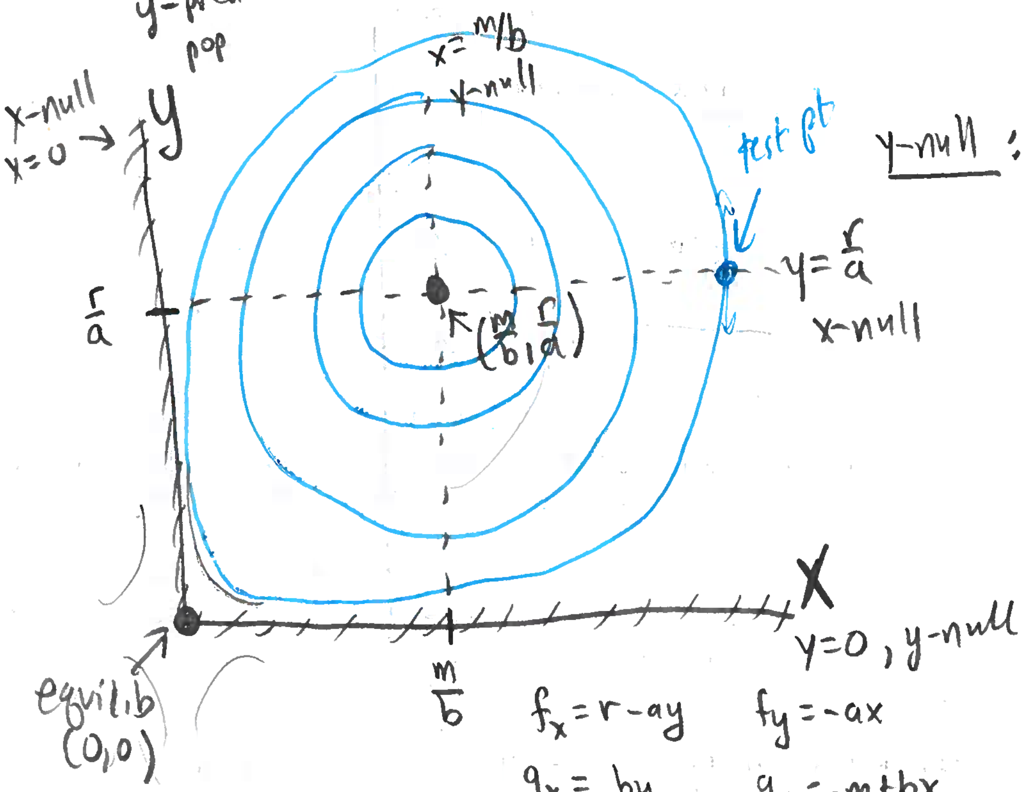
$x = 0$ $\quad$ $0 = r - ay$

$y = \frac{r}{a}$

y-null : $y' = 0 \rightarrow 0 = -my + bxy$

$= y(-m + bx)$

$y = 0$ $\quad$ $x = \frac{m}{b}$



$f_x = r - ay$ $\quad$ $f_y = -ax$

$g_x = by$ $\quad$ $g_y = -m + bx$

Jacobians

$$J(0,0) = \begin{array}{cc} x & y \\ \left(\begin{matrix} r & 0 \\ 0 & -m \end{matrix}\right) \end{array}$$

$\det J = -rm < 0$

$\rightarrow$ saddle

$$J\left(\frac{m}{b}, \frac{r}{a}\right) = \begin{pmatrix} 0 & -\frac{am}{b} \\ \frac{br}{a} & 0 \end{pmatrix}$$

$\det J = 0 - \left(-\frac{am}{b}\right)\left(\frac{br}{a}\right) = mr > 0$

$\operatorname{tr} J = 0$

$\Downarrow$

linearization has centers at equilibrium

$\vec{z}\,' = J\vec{z}$ (at (0,0))

$\Downarrow$

could have spirals or centers at original equilibrium $\left(\frac{m}{b}, \frac{r}{a}\right)$

How to see what happens?

Look at y-nullcline $x=\frac{m}{b}$

Divide the DE's:

$$\frac{x'}{y'} = \frac{dx/dt}{dy/dt} = \frac{rx-axy}{-my+bxy} = \frac{x(r-ay)}{y(-m+bx)}$$

$$\int \frac{-m+bx}{x}\,dx = \int \frac{r-ay}{y}\,dy$$

$$-m\ln(x)+bx+C = r\ln(y)-ay$$

↓ exp both sides

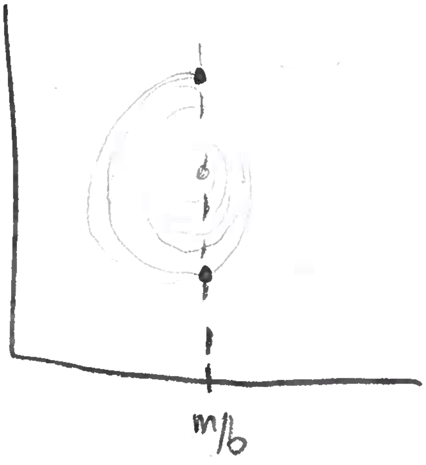
$$\tilde{C}\,x^{-m}e^{bx} = y^{r}e^{-ay}$$

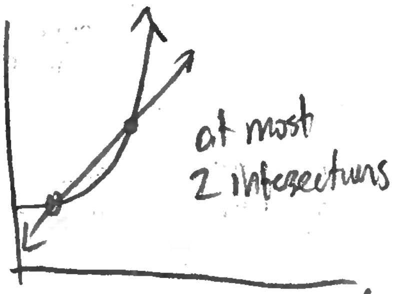
on nullcline → $x=\frac{m}{b}$

$$\underbrace{\tilde{C}\left(\frac{m}{b}\right)^{-m}e^{m}}_{\text{const } \hat{C}} = y^{r}e^{-ay} \Rightarrow \hat{C} = y^{r}e^{-ay}$$

$$\boxed{\hat{C}e^{ay} = y^{r}}$$

↓ Desmos





at most 2 intersections!

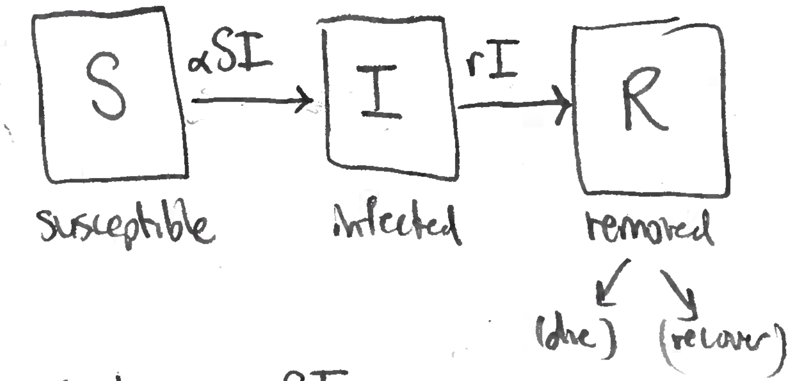
⇒ def not ∞-many ∩'s

⇒ not spiral

⇒ equilibrium $(\frac{m}{b}, \frac{r}{a})$ is a center!!

SIR

fix pop size $N = S+I+R \Rightarrow \boxed{R = N-S-I}$



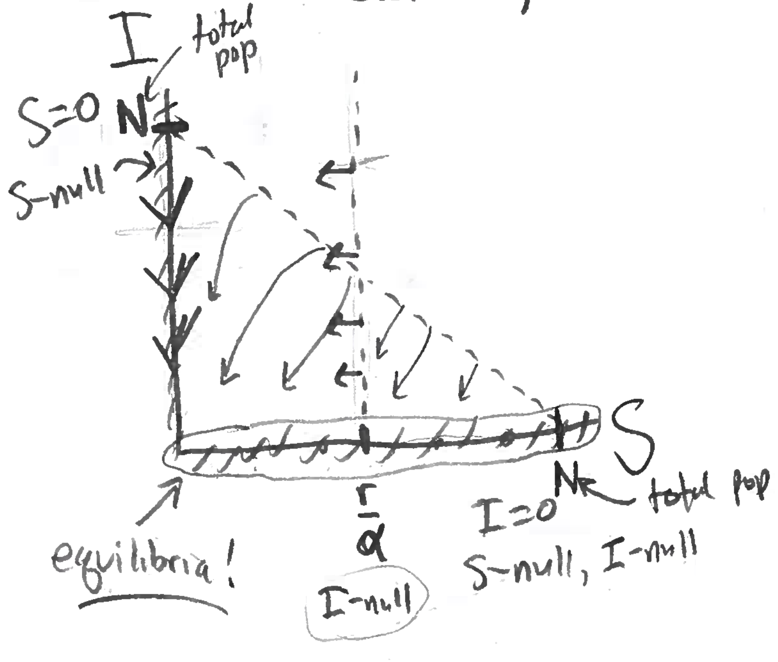
$(\alpha, r > 0)$

$$\begin{cases} S' = -\alpha SI \\ I' = \alpha SI - rI \\ S(0) = S_0,\ I(0) = I_0 \end{cases}$$

Find S-null $S' = 0$

$-\alpha SI = 0$

$S = 0,\ I = 0$



Find I-null $I' = 0$

$0 = I(\alpha S - r)$

$I = 0 \quad S = \frac{r}{\alpha}$

$$\frac{dI/dt}{dS/dt} = \frac{\alpha SI - rI}{-\alpha SI}$$

$$\frac{dI}{dS} = -1 + \frac{r}{\alpha S}$$

$$\int 1\, dI = \int -1 + \frac{r}{\alpha S}\, dS$$

$$I = -S + \frac{r}{\alpha}\ln(S) + C$$

$$I_0 = I(0) = -S(0) + \frac{r}{\alpha}\ln(S(0)) + C$$

$$C = \underbrace{I_0 + S_0}_{=N} - \frac{r}{\alpha}\ln(S_0) = N - \frac{r}{\alpha}\ln(S_0)$$

$$\Rightarrow I = -S + \frac{r}{\alpha}\ln(S) + N - \frac{r}{\alpha}\ln(S_0)$$