So far ⇝ only considered linear systems of eqts ($\vec{x}' = A\vec{x}$)
(2x2)

Now:

$$\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$$

Let $\vec{x}^* = (x_e, y_e)^T$ is a critical pt (equilibrium) of system,

meaning $\vec{x}^*$ solves $\begin{cases} 0 = f(x_e, y_e) \\ 0 = g(x_e, y_e) \end{cases}$

$\downarrow$ transform to linear system at each equilibrium

$$\boxed{\vec{z}' = J\vec{z}}, \text{ where}$$

$$J = \begin{pmatrix} f_x(x_e, y_e) & f_y(x_e, y_e) \\ g_x(x_e, y_e) & g_y(x_e, y_e) \end{pmatrix}$$

Side note: What is $f_x, f_y, g_x,$ and g_y?

Partial derivatives → calc 3 concept

$f(x,y)$
2 vars ~ each variable gets its own derivative called a partial derivative with respect to that variable: $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$

Main Fact: $\frac{\partial}{\partial x}$ treats "y" like a constant

$\frac{\partial}{\partial y}$ treats "x" like a constant

Compute

treat like constant

Ex:

$$\frac{\partial}{\partial x}\left[3x^2 y + x y^2\right] = 3y\,\frac{\partial}{\partial x}\left[x^2\right] + y^2\,\frac{\partial}{\partial x}\left[x\right]$$

$$= 6yx + y^2$$

$$\frac{\partial}{\partial y}\left[3x^2 y + x y^2\right] = 3x^2\,\frac{\partial}{\partial y}\left[y\right] + x\,\frac{\partial}{\partial y}\left[y^2\right]$$

$$= 3x^2 + 2xy$$

Ex: $\frac{\partial}{\partial x}\left[y^3\right] = 0 \qquad \frac{\partial}{\partial y}\left[x^3 + \sin(x)\right] = 0$

Notation for partial derivatives:

$$\frac{\partial f}{\partial x} \equiv f_x \qquad \frac{\partial f}{\partial y} \equiv f_y$$

Ex: $f(x,y) = x^3 y^4 + x y^5 + x^8 y^3$

$$\begin{cases} f_x = 3x^2 y^4 + y^5 + 8x^7 y^3 \\ f_y = 4x^3 y^3 + 5xy^4 + 3x^8 y^2 \end{cases}$$

$$f_x(1,1) = 3(1^2)(1^4) + 1^5 + 8(1^7)(1^3)$$

$$= 3 + 1 + 8 = 12$$

Behavior of soln to $\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$ near a crit pt

★ if $(0,0)$ is asymptotically stable in $\vec{z}\,' = J\vec{z}$,
then equilibrium it's based is asy stable in $\begin{cases} x' = f \\ y' = g \end{cases}$

~ occurs when J has $\ominus$ evals, OR
$\mathbb{C}$-valued evals w/ $\ominus$ real part

★ if $(0,0)$ unstable in $\vec{z}\,' = J\vec{z}$, then unstable
in $\begin{cases} x' = f \\ y' = g \end{cases}$ as well ~ at least one eval of J is $\oplus$
or is $\mathbb{C}$-valued with $\oplus$ real part

★ if $(0,0)$ center in $\vec{z}\,' = J\vec{z}$ ⇝ equilibrium of $\begin{cases} x' = f \\ y' = g \end{cases}$ may be center OR a spiral
(J has pure imaginary evals)

Nullclines: x-null : $x' = 0$ y-null : $y' = 0$
curves
~ $\cap$ of nullclines is always an equilibrium

How to sketch phase diagram

1. find crit pts / type, stability using $\vec{z}\,' = J\vec{z}$ at each crit pt
2. draw x-nullcline & y-nullcline
3. use DE to find flow informations (using test pts)
4. sketch orbits

f

make it nonlinear

Ex: $\begin{cases} x' = -x + xy \\ y' = -4y + 8xy \end{cases}$

g

Find equilibria (aka crit. pts): solve system

$$\begin{cases} 0 = -x + xy & \text{(i)} \\ 0 = -4y + 8xy & \text{(ii)} \end{cases}$$

$$\text{(i)} \rightarrow 0 = x(-1+y)$$

OR

$x = 0$	$y = 1$
(ii)	(ii)
$0 = -4y + 0$	$0 = -4(1) + 8x(1)$
$y = 0$	$0 = -4 + 8x$
	$x = \frac{1}{2}$
$(0,0)$	$(\frac{1}{2}, 1)$

Crit pts

Form Jacobians:

$f_x = -1 + y \qquad f_y = x$

$g_x = 8y \qquad g_y = -4 + 8x$

$$\text{at } (\overset{x}{0}, \overset{y}{0})$$

$$J = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix}$$

$$\Downarrow$$

$$\vec{z}\,' = J\vec{z}$$

$$\det J = (-1)(-4) - 0 = 4 > 0$$

$$\operatorname{tr} J = -1 + (-4) = -5$$

$$\Downarrow$$

Thm 4.40: Stable node

(note: evals here are $\lambda = -1, -4$)

$$\text{at } (\overset{x}{1/2}, \overset{y}{1})$$

$$J = \begin{pmatrix} 0 & 1/2 \\ 8 & 0 \end{pmatrix}$$

$$\Downarrow$$

$$\det J = 0 - 4 < 0$$

$$\Downarrow$$

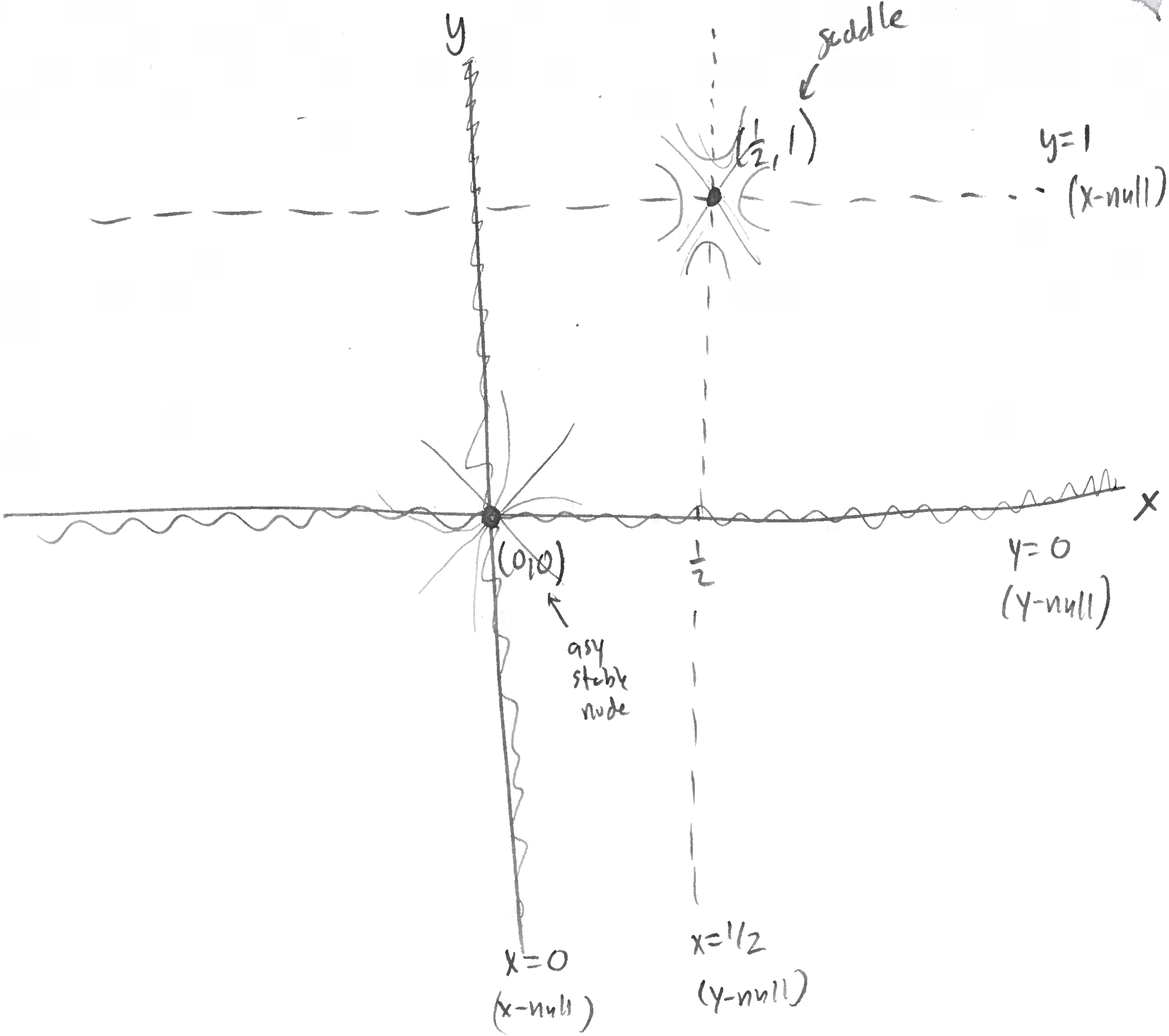
Saddle (unstable)

(evals: $\lambda = \pm 2$)

<u>nullclines</u>:

<u>x-null</u>: $0 = -x + xy \rightarrow 0 = x(-1+y) \rightarrow x = 0,\ y = 1$ x-null

<u>y-null</u>: $0 = -4y + 8xy \rightarrow 0 = y(-4+8x) \rightarrow y = 0,\ x = 1/2$ y-null


y
saddle
$(\frac{1}{2}, 1)$
y=1
(x-null)
x
(0,0)
$\frac{1}{2}$
y=0
(y-null)
asy
stable
node
x=0
(x-null)
x=1/2
(y-null)

Finish Wed!