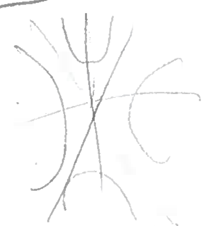
§4.5 Phase plane analysis

Given $\vec{x}\,' = A\vec{x}$, we get characteristic eqt

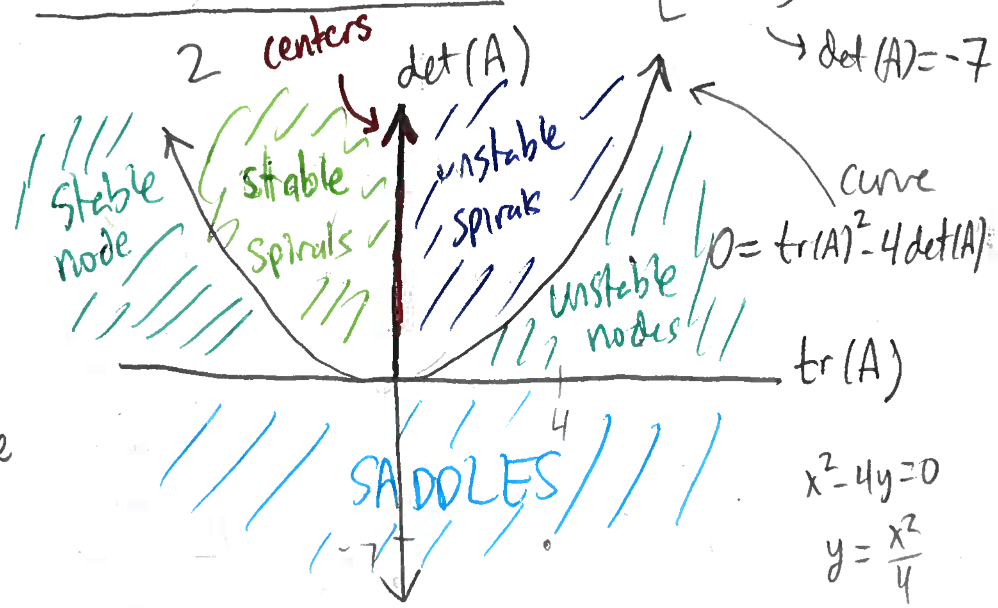
$$\lambda^2 - (\operatorname{tr}(A))\lambda + \det(A) = 0$$

$$\operatorname{tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a + d$$

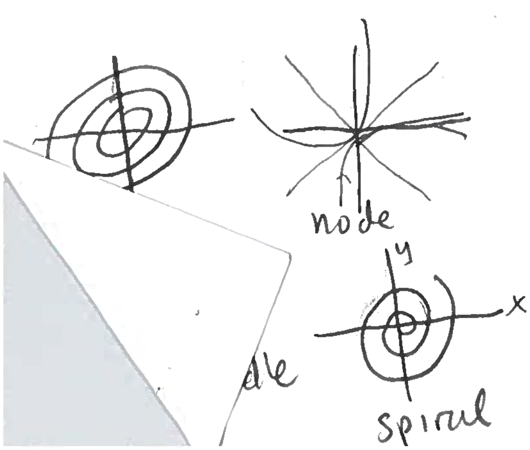


$$\lambda = \frac{\operatorname{tr}(A) \pm \sqrt{\operatorname{tr}(A)^2 - 4\det(A)}}{2}$$

$A = \begin{bmatrix} -1 & 1 \\ 2 & 5 \end{bmatrix}$ → $\operatorname{tr}(A) = 4$, → $\det(A) = -7$



eigenvalue	orbit
$\lambda_1, \lambda_2 > 0$ real unequal	unstable node
$\lambda_1, \lambda_2 < 0$ real unequal	asymptotically stable node
$\lambda_1 < 0 < \lambda_2$ (opp signs)	(unstable) saddle
$\lambda = \pm bi$ (pure imaginary)	centers
$\lambda = a \pm bi$, $a > 0$	spirals ~ unstable
$\lambda = a \pm bi$, $a < 0$	~~u~~stable spirals
$\lambda_1 = \lambda_2 > 0$	unstable node
$\lambda_1 = \lambda_2 < 0$	stable node



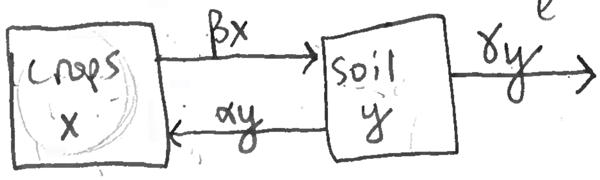
Theorem 4.40

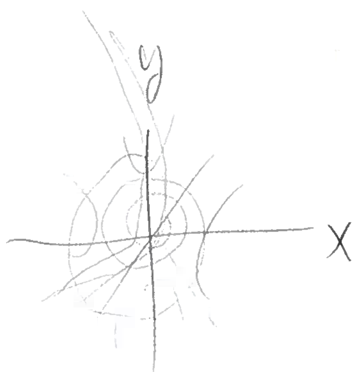
det A	tr A	orbit
<0	any	saddle
>0	$=0$	centers
>0	>0	unstable node if $4\det A < \operatorname{tr} A$
>0	>0	unstable spiral if ~ > ~
>0	<0	asy stable node if ~ ≤ ~
>0	<0	~~a~~ spiral if ~ > ~

Ex 4.42 (Crop-soil) (See 19 March notes)

$\beta, \alpha, \gamma > 0$

pesticide that is destroyed from soil





$$\vec{x}\begin{cases} x' = -\beta x + \alpha y \\ y' = \beta x - (\alpha+\gamma) y \end{cases} \longrightarrow \vec{x}' = \begin{pmatrix} -\beta & \alpha \\ \beta & -(\alpha+\gamma) \end{pmatrix} \vec{x}$$

Notice: ① $\operatorname{tr}\begin{pmatrix} -\beta & \alpha \\ \beta & -(\alpha+\gamma) \end{pmatrix} = -\beta - \alpha - \gamma < 0$

$\Rightarrow$ phase plane analysis $\rightarrow$ either stable or a saddle orbit

② $\det\begin{pmatrix} -\beta & \alpha \\ \beta & -(\alpha+\gamma) \end{pmatrix} = \beta(\cancel{\alpha}+\gamma) - \cancel{\alpha\beta} = \beta\gamma > 0$

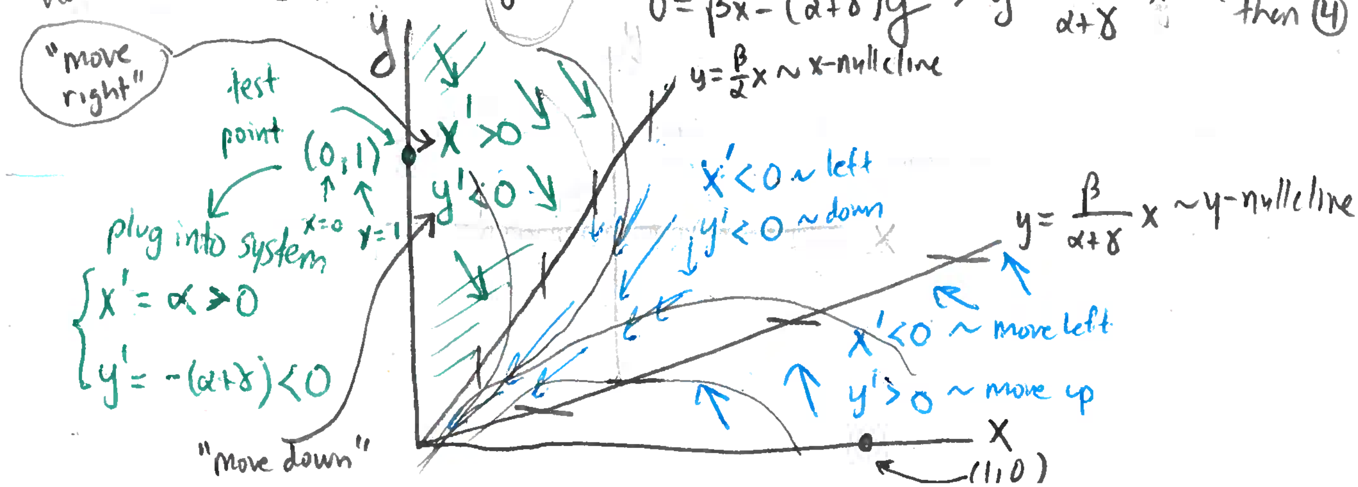
$\Rightarrow$ combined w/ ① the orbit is either a stable node or stable spiral

③ Quadrant I of xy-plane is relevant (negative pesticide makes no sense)

orbit is vertical here $\rightarrow$ ④ x-nullcline: $x' = 0 \rightarrow 0 = -\beta x + \alpha y$

$y = \frac{\beta}{\alpha} x$

orbit is horizontal $\rightarrow$ ⑤ y-nullcline: $y' = 0 \rightarrow 0 = \beta x - (\alpha+\gamma) y \rightarrow y = \frac{\beta}{\alpha+\gamma} x$. (smaller slope than ④)



Ex: $\begin{cases} x' = -7x + 6y \\ y' = 6x + 2y \end{cases} \rightsquigarrow A = \begin{pmatrix} -7 & 6 \\ 6 & 2 \end{pmatrix}$

$\det A = -14 - 36 < 0$

$\downarrow$

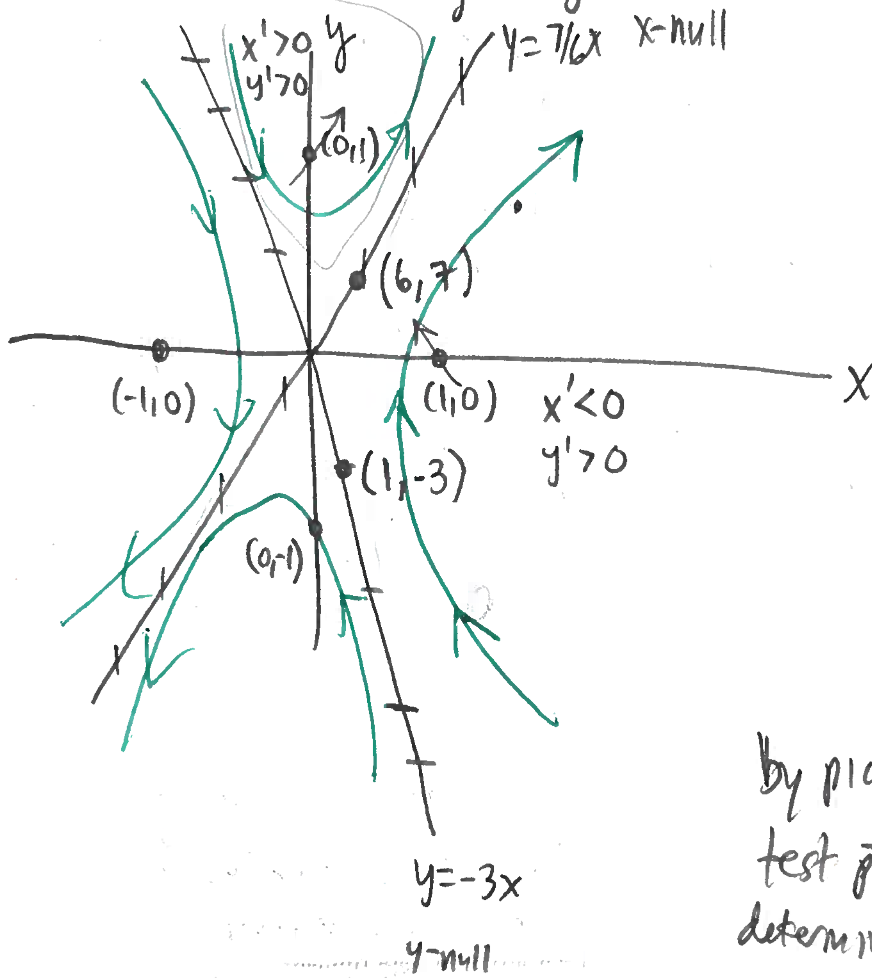
saddle

x-nullcline: $x' = 0$

$0 = -7x + 6y \to y = \frac{7}{6}x$

y-nullcline: $y' = 0$

$0 = 6x + 2y \to y = -3x$



by picking test pts, can determine orbits