Ex: Solve

$$\vec{x}' = \begin{pmatrix} 1 & -2 \\ 3 & -1 \end{pmatrix} \vec{x}$$

eigenvalues: $\det \begin{pmatrix} 1-\lambda & -2 \\ 3 & -1-\lambda \end{pmatrix} = 0$

$$\underbrace{(-1-\lambda)(1-\lambda)} - (-6) = 0$$

$$-1 + \cancel{\lambda} - \cancel{\lambda} + \lambda^2 + 6 = 0$$

$$\lambda^2 + 5 = 0 \longrightarrow \lambda = \pm\sqrt{5}\,i = \underset{\substack{\uparrow \\ a}}{0} \pm \underset{\substack{\uparrow \\ b}}{\sqrt{5}}\,i$$

eigenvector for $\lambda = \sqrt{5}\,i$:

$(1-\sqrt{5}i)(1+\sqrt{5}i)$

$= 1+5 = 6$

$\Downarrow$

$\text{row } 2 = \frac{(\text{row } 1)(1+\sqrt{5}i)}{2}$

$$\begin{pmatrix} 1-\sqrt{5}i & -2 \\ 3 & -1-\sqrt{5}i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Downarrow$$

$$(1-\sqrt{5}i)v_1 - 2v_2 = 0$$

$$v_2 = \frac{1-\sqrt{5}i}{2} v_1 \implies \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ \frac{1}{2}(1-\sqrt{5}i)v_1 \end{pmatrix}$$

$$= v_1 \begin{pmatrix} 1 \\ \frac{1}{2} - \frac{\sqrt{5}}{2} i \end{pmatrix}$$

Therefore by 5 April notes' technique:

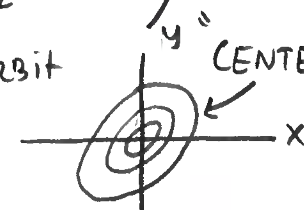
$$= v_1 \left[\underset{\substack{\uparrow \\ \vec{w}}}{\begin{pmatrix} 1 \\ 1/2 \end{pmatrix}} + \underbrace{\begin{pmatrix} 0 \\ -\sqrt{5}/2 \end{pmatrix}}_{\vec{z}} i \right]$$

$$\vec{x}(t) = c_1 e^{0t} \underbrace{\left[\begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \cos(\sqrt{5}t) - \begin{pmatrix} 0 \\ -\sqrt{5}/2 \end{pmatrix} \sin(\sqrt{5}t) \right]}_{\vec{x}_1} + c_2 e^{0t} \left[\begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \sin(\sqrt{5}t) + \begin{pmatrix} 0 \\ -\sqrt{5}/2 \end{pmatrix} \cos(\sqrt{5}t) \right]$$

$$= c_1 \begin{pmatrix} \cos(\sqrt{5}t) \\ \frac{1}{2}\cos(\sqrt{5}t) + \sin(\sqrt{5}t) \end{pmatrix} + c_2 \begin{pmatrix} \sin(\sqrt{5}t) \\ \frac{1}{2}\sin(\sqrt{5}t) - \frac{\sqrt{5}}{2}\cos(\sqrt{5}t) \end{pmatrix}$$

$$= \begin{pmatrix} c_1 \cos(\sqrt{5}t) + c_2 \sin(\sqrt{5}t) \\ \left(\frac{c_1}{2} - \frac{c_2\sqrt{5}}{2}\right)\cos(\sqrt{5}t) + \left(c_1 + \frac{c_2}{2}\right)\sin(\sqrt{5}t) \end{pmatrix} \longrightarrow$$

orbit



"CENTER"

p. 221 #1a eigenpair: $\lambda = \pm i$

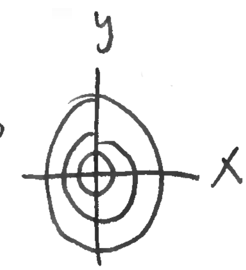
$$\vec{v} = \begin{pmatrix} 1 \\ \pm i \end{pmatrix}$$

Use $\lambda = i$, $\vec{v} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{w}} + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{z}} i$

$= \underset{a}{0} + \underset{b=1}{i}$

Soln is

$$\vec{x}(t) = c_1 e^{0t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos(t) - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin(t)\right)$$
$$+ c_2 e^{0t}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos(t)\right)$$
$$= \begin{pmatrix} c_1\cos(t) + c_2\sin(t) \\ -c_1\sin(t) + c_2\cos(t) \end{pmatrix}$$



§4.4.5 Real, equal evals ~ $\lambda_1 = \lambda = \lambda_2$

$\left(\vec{x}\,' = A\vec{x}\right)$

Two cases:

Case 1

λ has two indep evects $\vec{v}_1, \vec{v}_2$ (not multiples)

$\Downarrow$

$\vec{x} = c_1\vec{v}_1 e^{\lambda t} + c_2\vec{v}_2 e^{\lambda t}$

Case 2

only one evector $\vec{v}$

("deficient matrix")

$\downarrow$

$\vec{x}_1 = c_1\vec{v}e^{\lambda t}$ ✓

But what abt 2^{nd} soln?

Trick: guess $\vec{x}_2 = (t\vec{v} + \vec{w})e^{\lambda t}$

known, unknown, known

$\vec{v}$ is an eigenvector ... w/ eval λ

Meaning: $A\vec{v} = \lambda\vec{v}$

LHS of DE $\longrightarrow$ $\vec{x}_2\,' = \vec{v}e^{\lambda t} + (t\vec{v} + \vec{w})\lambda e^{\lambda t}$

product rule

RHS of DE $\longrightarrow$ $A\vec{x}_2 = A\left[(t\vec{v} + \vec{w})e^{\lambda t}\right] = e^{\lambda t}\left[t\boxed{A\vec{v}} + A\vec{w}\right]$ (scalar)

$= e^{\lambda t}\left[t\lambda\vec{v} + A\vec{w}\right]$

to be a soln

$\vec{v}e^{\lambda t} + \cancel{t\lambda\vec{v}e^{\lambda t}} + \vec{w}\lambda e^{\lambda t} = \vec{x}_2\,' = A\vec{x}_2 = \cancel{t\lambda\vec{v}e^{\lambda t}} + A\vec{w}e^{\lambda t}$

$\downarrow$ div by $e^{\lambda t}$

$\vec{v} + \lambda\vec{w} = A\vec{w}$

$\boxed{\vec{v} = (A - \lambda I)\vec{w}}$ $\Longrightarrow$ solve for $\vec{w}$ $\Rightarrow$ get soln

$\vec{w}$ generalized eigenvector

$$\boxed{\vec{x}(t) = c_1\vec{v}e^{\lambda t} + c_2 e^{\lambda t}(t\vec{v} + \vec{w})}$$

Ex 4.3.8 : $\vec{x}' = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \vec{x}$

evals: $\det \begin{pmatrix} -2-\lambda & 0 \\ 0 & -2-\lambda \end{pmatrix} = 0$

$(-2-\lambda)^2 = 0$

$-2-\lambda = 0$

$\lambda = -2$ double root

eigenvector:

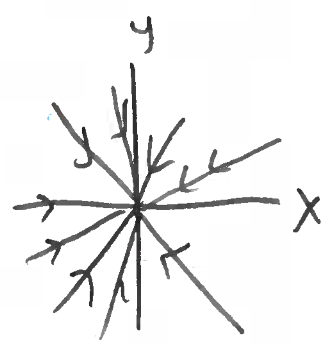
$\begin{pmatrix} -2-(-2) & 0 \\ 0 & -2-(-2) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Wow! ~ means ~ no restrictions on v_1, v_2 .... completely free to choose!

easiest: $\vec{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\vec{v_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\vec{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t} = \begin{pmatrix} c_1 e^{-2t} \\ c_2 e^{-2t} \end{pmatrix}$



stable node

Ex 4.3.9:

$$\vec{x}' = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \vec{x}$$

$$\det\begin{pmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{pmatrix} = 0 \rightarrow (2-\lambda)(4-\lambda)+1=0$$

$$8-2\lambda-4\lambda+\lambda^2+1=0$$

$$\lambda^2-6\lambda+9=0$$

$$(\lambda-3)^2=0$$

$\lambda=3$ double root

evector

$$\begin{pmatrix} 2-3 & 1 \\ -1 & 4-3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-v_1+v_2=0 \rightarrow \boxed{v_2=v_1} \rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = v_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\Rightarrow$ eigenpair $\boxed{\lambda=3;\ \vec{v}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

Let $\vec{x_2} = (t\vec{v}+\underset{\uparrow\ \text{unknown}}{\vec{w}})e^{\lambda t} \rightsquigarrow$ solve

$$\underset{\vec{v}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\underset{\vec{w}}{\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}}$$

$$\rightarrow -w_1+w_2=1$$

$$w_2=1+w_1$$

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ 1+w_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \underset{\uparrow\ \text{free var}}{w_1}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

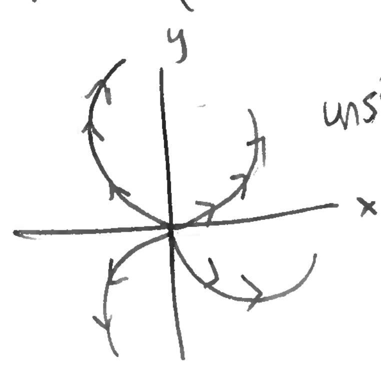
$$\underset{w_1=1}{\Rightarrow} \vec{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

$$\vec{x_2} = \left(\underbrace{t\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}}\right)e^{3t}$$

Soln is

$$\vec{x}(t) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{3t} + c_2\begin{pmatrix} t+1 \\ t+2 \end{pmatrix}e^{3t}$$

$$= \begin{pmatrix} c_1e^{3t} + c_2(t+1)e^{3t} \\ c_1e^{3t} + c_2(t+2)e^{3t} \end{pmatrix}$$



unstable node