1

Ex: $\vec{x}\,' = \overbrace{\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}}^{A}\vec{x}$

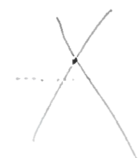
$\det A = (1)(4)-(-2)(-2)$
$= 4-4 = 0$

$\vec{x}\,' = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

line of equilibria: $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$x - 2y = 0$

$\boxed{y = \frac{x}{2}}$ ← our line of equilibria



eigenvalues: $\det(A - \lambda I) = 0$

$\det\begin{pmatrix} 1-\lambda & -2 \\ -2 & 4-\lambda \end{pmatrix} = 0$

$(1-\lambda)(4-\lambda) - (-2)(-2) = 0$

$\cancel{4} - 5\lambda + \lambda^2 - \cancel{4} = 0$

$\lambda(\lambda - 5) = 0$

$\lambda = 0, 5$

evectors

$\lambda_1 = 0$

$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\boxed{v_1 = 2v_2}$

$2v_2$
v_2

eigenpair $\boxed{\lambda_1 = 0,\ \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$

$\lambda_2 = 5$

$\begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

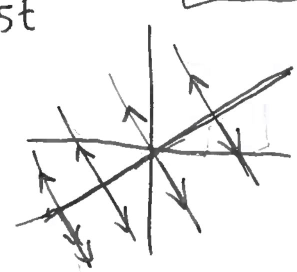
$-4v_1 - 2v_2 = 0 \rightarrow v_2 = -2v_1$

$\vec{v}_2 = \begin{pmatrix} v_1 \\ -2v_1 \end{pmatrix} = v_1\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

eigenpair $\boxed{\lambda_2 = 5,\ \vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$

gen soln: $\vec{x}(t) = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix}e^{0t} + c_2\begin{pmatrix} 1 \\ -2 \end{pmatrix}e^{5t}$ ($e^{0t} \to 1$)

$= \begin{pmatrix} 2c_1 + c_2e^{5t} \\ c_1 - 2c_2e^{5t} \end{pmatrix}$



equilibria

det A=0 case
unstable

§4.4.2 Complex eigenvalues

Recall: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ (Euler's formula)

If evalues are of form $a \pm bi$

corresp eigenvectors: $\vec{v} = \vec{w} + i\vec{z}$ (here: $\vec{w}$ and $\vec{z}$ have no "i" in them)

Suppose we get eigenpair

$$\Big(\underbrace{a+bi}_{\lambda},\ \underbrace{\vec{w}+i\vec{z}}_{\vec{v}}\Big)$$

Soln will be

$e^{a+bi} = e^{at}e^{bit}$

$$\vec{x} = \vec{v}e^{\lambda t} = (\vec{w}+i\vec{z})e^{(a+bi)t}$$
$$= e^{at}(\vec{w}+i\vec{z})(\cos(bt)+i\sin(bt))$$
$$= e^{at}[\vec{w}\cos(bt) + \vec{w}i\sin(bt) + i\vec{z}\cos(bt) - \vec{z}\sin(bt)]$$
$$= \underbrace{e^{at}[\vec{w}\cos(bt) - \vec{z}\sin(bt)]}_{\vec{x}_1} + i\underbrace{e^{at}[\vec{w}\sin(bt) + \vec{z}\cos(bt)]}_{\vec{x}_2}$$

indep solns

So we get general soln

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2$$

Consequence: only need to find ONE eigenvector to get both solns!!

Ex 4.37 $\quad \vec{x}' = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \vec{x}$

evals: $\det \begin{pmatrix} -2-\lambda & -3 \\ 3 & -2-\lambda \end{pmatrix} = 0$

$$(2-\lambda)^2 + 9 = 0$$
$$(2-\lambda)^2 = -9$$
$$2-\lambda = \pm\sqrt{-9} = \pm 3i$$
$$\lambda = 2 \mp 3i$$

$\boxed{\lambda = 2+3i} \rightarrow$ find e-vector:

$$\begin{pmatrix} -2-(2+3i) & -3 \\ 3 & -2-(2+3i) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -3i & -3 \\ 3 & -3i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-3iv_1 - 3v_2 = 0 \longrightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ -iv_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$v_2 = \frac{-3i}{3} v_1 = -iv_1$

$\Rightarrow$ eigenpair: $\lambda = \overset{a}{2} + \overset{b}{3}i$

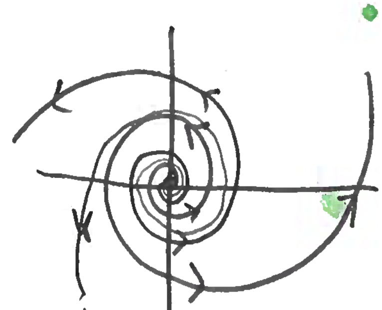
$$\vec{v} = \begin{pmatrix} 1 \\ -i \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{w}} + i \underbrace{\begin{pmatrix} 0 \\ -1 \end{pmatrix}}_{\vec{z}}$$

So gen soln is

$$\vec{x}(t) = c_1 e^{2t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos(3t) - \begin{pmatrix} 0 \\ -1 \end{pmatrix}\sin(3t)\right] + c_2 e^{2t}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin(3t) + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\cos(3t)\right]$$

$$= c_1 e^{2t}\begin{pmatrix} \cos(3t) \\ \sin(3t) \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} \sin(3t) \\ -\cos(3t) \end{pmatrix}$$

$$= \begin{pmatrix} c_1 e^{2t}\cos(3t) + c_2 e^{2t}\sin(3t) \\ c_1 e^{2t}\sin(3t) - c_2 e^{2t}\cos(3t) \end{pmatrix}$$



unstable spiral