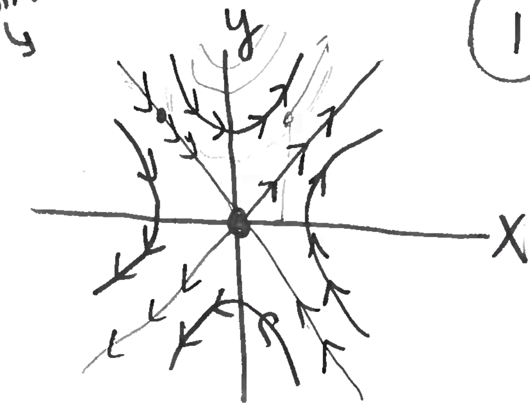
Desmos




SADDLE

① 

**Ex**: Solve $\begin{cases} x' = x+y \\ y' = 4x+y \end{cases}$

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \vec{x}\,' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x}$$

$x(0)=1$

$y(0)=2$

eigenvalues of $\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$:

$$0 = \det \begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)^2 - 4$$

$$4 = (1-\lambda)^2 \longrightarrow 1-\lambda = \pm 2$$

$$\lambda = 1 \mp 2 \rightarrow \boxed{\lambda = -1, 3}$$

eigenvalues

**eigenvectors**

$\underline{\lambda_1 = -1}$

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2v_1 + v_2 = 0$$

$$\boxed{v_2 = -2v_1}$$

So, $\vec{v_1} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ -2v_1 \end{pmatrix} = v_1 \underbrace{\begin{pmatrix} 1 \\ -2 \end{pmatrix}}$

note: this is a different thing than "$v_1$" above

$\Rightarrow$ eigenpair

$$\boxed{\lambda_1 = -1, \ \vec{v_1} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$

$\underline{\lambda_2 = 3}$

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-2v_1 + v_2 = 0$$

$$v_2 = 2v_1$$

So $\vec{v_2} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ 2v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\Rightarrow$ eigenpair $\boxed{\lambda_2 = 3, \ \vec{v_2} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$

$\Rightarrow$ general soln

$$x(t) = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} = \begin{pmatrix} c_1 e^{-t} + c_2 e^{3t} \\ -2c_1 e^{-t} + 2c_2 e^{3t} \end{pmatrix}$$

x

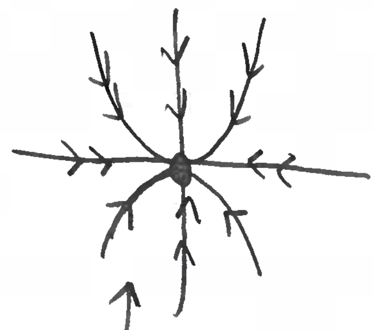
y

Ex: $\begin{cases} x' = -x \\ y' = \quad -2y \end{cases}$ ← "decoupled"
can solve w/o matrices

$x' = -x \to x = e^{-t}$

$y' = -2y \to y = e^{-2t}$

$\Rightarrow$ gen soln $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{-2t} \end{pmatrix}$ ← x, ← y



asymptotically stable node

notice: $y = c_2 e^{-2t} \to e^{-2t} = \frac{y}{c_2}$

$x^2 = c_1^2 e^{-2t}$

$= \frac{c_1^2}{c_2} y \to \boxed{y = \frac{c_2}{c_1^2} x^2}$

Ex: $\begin{cases} x' = -2x + 2y \\ y' = 2x - 5y \end{cases} \to \vec{x}' = \begin{pmatrix} -2 & 2 \\ 2 & -5 \end{pmatrix} \vec{x} \to$ evals: $\det \begin{pmatrix} -2-\lambda & 2 \\ 2 & -5-\lambda \end{pmatrix} = 0$

$(-2-\lambda)(-5-\lambda) - 4 = 0$

$10 + 7\lambda + \lambda^2 - 4 = 0$

$\lambda^2 + 7\lambda + 6 = 0$

$(\lambda+6)(\lambda+1) = 0 \to \lambda = -6, -1$

eigvects

$\lambda_1 = -6$

$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$2v_1 + v_2 = 0$

$v_2 = -2v_1$

$\downarrow$

$\vec{v}_1 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ -2v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\downarrow$

$\boxed{\lambda_1 = -6, \vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$ → gen soln

$\lambda_2 = -1$

$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

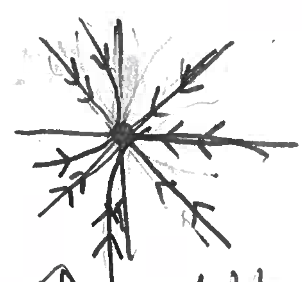
$-v_1 + 2v_2 = 0$

$v_1 = 2v_2 \rightsquigarrow \vec{v}_2 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2v_2 \\ v_2 \end{pmatrix} = v_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\downarrow$

$\boxed{\lambda_2 = -1, \vec{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$

$\downarrow$



asy stable node

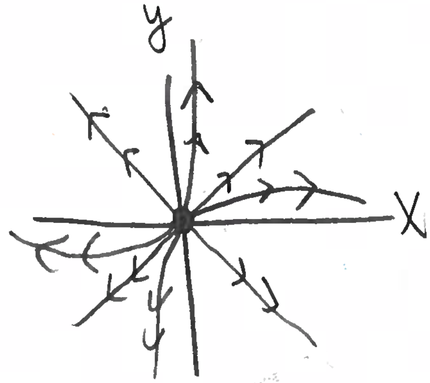
gen soln

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-6t} + c_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} = \begin{pmatrix} c_1 e^{-6t} + 2c_2 e^{-t} \\ -2c_1 e^{-6t} + c_2 e^{-t} \end{pmatrix}$$

**Ex:** Both evals $\oplus$ $\rightsquigarrow$ unstable node

e.g. soln like

$$\vec{x}(t) = \begin{pmatrix} c_1 e^{t} + c_2 e^{4t} \\ 2c_1 e^{t} - 2c_2 e^{4t} \end{pmatrix}$$




$\begin{pmatrix} a \\ b \end{pmatrix}$ slope $\frac{b}{a}$

**Ex:** The case $\det A = 0$

Here, equilibria lie on a line (as we saw earlier)

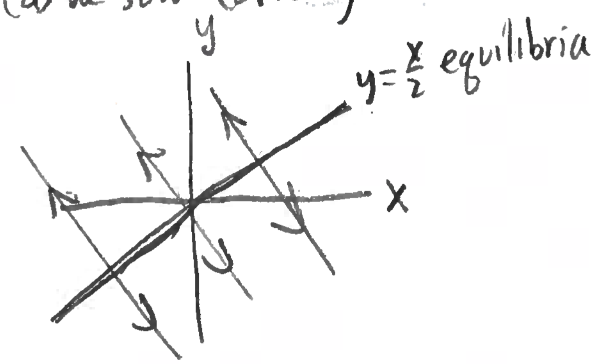
$$\begin{cases} x' = x - 2y \\ y' = -2x + 4y \end{cases} \rightarrow \vec{x}\,' = \underbrace{\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}}_{\det = 0} \vec{x}$$

$\Downarrow$

line of equilibria

$x - 2y = 0$

$y = \frac{x}{2}$




evals

$$\det \begin{pmatrix} 1-\lambda & -2 \\ -2 & 4-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(4-\lambda) - 4 = 0$$

$$\cancel{4} - 5\lambda + \lambda^2 - \cancel{4} = 0$$

$$\lambda(\lambda - 5) = 0 \rightarrow \lambda = 0, 5$$

$$\vec{x} = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{\overset{1}{\downarrow}\, 0t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{5t}$$

$$= \begin{pmatrix} 2c_1 + c_2 e^{5t} \\ c_1 - 2c_2 e^{5t} \end{pmatrix}$$

evects

$$\lambda_1 = 0: \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow v_1 = 2v_2 \rightarrow \boxed{\lambda_1 = 0,\ \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

$$\lambda_2 = 5: \begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \underset{v_2 = -2v_1}{-2v_1 - v_2 = 0} \rightarrow \boxed{\lambda_2 = 5,\ \vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$$