Summary: To solve $A\vec{v}=\lambda\vec{v}$

① Solve $\det(A-\overset{\text{unknown}}{\boxed{\lambda}}I)=0$, i.e.

$$\lambda^2-tr(A)\lambda+\det(A)=0$$

for λ to find evals

② plug in evalues into $(A-\lambda I)\overset{\text{unknown}}{\boxed{\vec{v}}}=\vec{0}$ to find evectors

Two facts

① Any const multiple of an eigenvector is still an evector:

if $\underbrace{A\vec{v}=\lambda\vec{v}}_{\vec{v}\text{ is an evec}}$, then $\underbrace{A(c\vec{v})=c(A\vec{v})=c(\lambda\vec{v})=\lambda(c\vec{v})}_{\text{shows } c\vec{v} \text{ is an evect.}}$

② if $\underbrace{\lambda_1\neq\lambda_2}_{\text{two evals}}$, then their corresponding evectors are independent (not multiples)

Ex 4.27: $A=\begin{pmatrix}1&1\\4&1\end{pmatrix}$ $\qquad I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$

Solve eigenvalues: $0=\det(A-\lambda I)$

$$0=\det\left(\begin{pmatrix}1&1\\4&1\end{pmatrix}-\begin{pmatrix}\lambda&0\\0&\lambda\end{pmatrix}\right)$$

$$0=\det\begin{pmatrix}1-\lambda&1\\4&1-\lambda\end{pmatrix}=(1-\lambda)^2-4=\lambda^2-2\lambda-3$$

$$=(\lambda-3)(\lambda+1)$$

$\Rightarrow$ Solutions

$\lambda_1=3$ and $\lambda_2=-1$

$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$

eigenvectors:

Plug into

$(A - \lambda I)\vec{v} = \vec{0}$

$\lambda_1 = 3$

Plug in:

$$\left(\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\downarrow$$

$$\begin{cases} -2v_1 + v_2 = 0 & (i) \\ 4v_1 - 2v_2 = 0 & (ii) \end{cases}$$

multiples of each other

⇓

only one matters

$\downarrow$ (i) solve for v_2

$$\boxed{v_2 = 2v_1}$$

$\downarrow$

eigenvector is

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ 2v_1 \end{pmatrix} = v_1\underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

usually think of this as the evec

$\downarrow$

eigenpair: $\boxed{\begin{matrix} \lambda_1 = 3 \\ \vec{v_1} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{matrix}}$

$\lambda_2 = -1$

Plug in:

$$\left(\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - (-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\downarrow$$

$$\begin{cases} 2v_1 + v_2 = 0 & (i) \\ 4v_1 + 2v_2 = 0 & (ii) \end{cases}$$

multiples

↓

only one matters

$\downarrow$ solve (i) for v_2

$$\boxed{v_2 = -2v_1}$$

$\downarrow$

eigenvector is

$$\vec{v_2} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ -2v_1 \end{pmatrix} = v_1\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$\downarrow$

eigenpair:

$$\boxed{\begin{matrix} \lambda_2 = -1 \\ \vec{v_2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \end{matrix}}$$

Ex 4.28 $A=\begin{pmatrix}-2 & -3\\ 3 & -2\end{pmatrix}$

$$\begin{array}{r}4\ 12\\ \cancel{52}\\ -16\\ \hline 36\end{array}$$

$$\begin{array}{r}13\\ 4\\ \hline 5\ 2\end{array}$$

16-52

evals: $\det(A-\lambda I)=0$

$$\det\begin{pmatrix}-2-\lambda & -3\\ 3 & -2-\lambda\end{pmatrix}=0$$

$(-2-\lambda)^2$

$\shortparallel$

$(\lambda+2)^2$

$$(-2-\lambda)^2+9=0$$

$$\lambda^2+4\lambda+4+9=0$$

$$\lambda^2+4\lambda+13=0 \xrightarrow{QF} \lambda=\frac{-4\pm\sqrt{4^2-4(13)}}{2}$$

$$=-2\pm\frac{1}{2}\sqrt{-36}$$

$$=-2\pm 3i$$

Eigevetors:

$\lambda_1=-2+3i$

$$(A-\lambda I)\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}-3i & -3\\ 3 & -3i\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$\downarrow$

$$-3iv_1-3v_2=0$$

$\downarrow$

$$v_2=-iv_1$$

$\downarrow$

$\lambda_1=-2+3i$

evec

$$\vec{v}_1=\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}v_1\\ -iv_1\end{pmatrix}=v_1\begin{pmatrix}1\\ -i\end{pmatrix}$$

$\lambda_2=-2-3i$

$$\begin{pmatrix}3i & -3\\ 3 & 3i\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$\downarrow$

$$3iv_1-3v_2=0$$

$$v_2=iv_1 \rightarrow \vec{v}_2=\begin{pmatrix}v_1\\ v_2\end{pmatrix}=v_1\begin{pmatrix}1\\ i\end{pmatrix}$$

$\downarrow$

$$\begin{cases}\lambda_2=-2+3i\\ v_2=\begin{pmatrix}1\\ i\end{pmatrix}\end{cases}$$

p. 207 #2a Find evals of $\begin{pmatrix} 2 & \beta \\ -1 & 0 \end{pmatrix}$ in terms of β.

Look at

$$\det\left(\begin{pmatrix} 2 & \beta \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = 0$$

$$\det\begin{pmatrix} 2-\lambda & \beta \\ -1 & -\lambda \end{pmatrix} = 0$$

$$-2\lambda + \lambda^2 + \beta = 0$$

$$\lambda^2 - 2\lambda + \beta = 0$$

$$\Big\downarrow \text{QF}$$

$$\lambda = \frac{2 \pm \sqrt{4-4\beta}}{2}$$

$\beta < 1 \rightsquigarrow$ 2 distinct real roots

$\beta = 1 \rightsquigarrow$ repeated root (same eval twice)

$\beta > 1 \rightsquigarrow$ complex roots

§4.4 Solving $\vec{x}\,' = A\vec{x}$

We already know

① if $\det A \neq 0$, then $x^* = \begin{pmatrix}0\\0\end{pmatrix}$ is only equilibrium

② if $\det A = 0$, then a line of equilibria

③ if $(\lambda_i; \vec{v}_i)$ is an eigenpair, then $c_i \vec{v}_i e^{\lambda_i t}$ solves system

§4.4.1 Real, unequal evals

3 possibilities

(i) λ_1, λ_2 ~ opp signs

(ii) ~ both ⊕

(iii) ~ both ⊖

Ex: $\vec{x}\,' = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\vec{x}$

evals: $0 = \det\begin{pmatrix}-1-\lambda & 0\\ 0 & 1-\lambda\end{pmatrix} = (-1-\lambda)(1-\lambda)$

$\lambda_1 = -1$

$\begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

row 2: $\boxed{v_2 = 0}$

↓

epair: $\boxed{\lambda_1 = -1,\ \vec{v}_1 = \begin{pmatrix}1\\0\end{pmatrix}}$

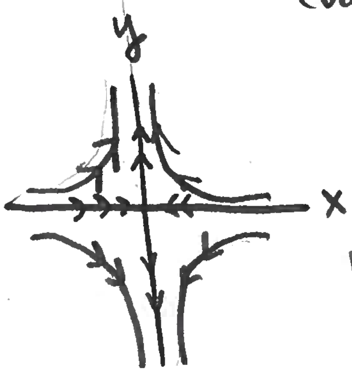
$\lambda_2 = 1$

$\begin{pmatrix}-2 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$-2v_1 = 0 \to v_1 = 0$

↓

epair: $\boxed{\lambda_2 = 1,\ \vec{v}_2 = \begin{pmatrix}0\\1\end{pmatrix}}$



SADDLE

$\Rightarrow$ general soln is $\vec{x}(t) = c_1\begin{pmatrix}1\\0\end{pmatrix}e^{-t} + c_2\begin{pmatrix}0\\1\end{pmatrix}e^{t} = \begin{pmatrix}c_1 e^{-t}\\ c_2 e^{t}\end{pmatrix}\begin{matrix}\to x\\ \leftarrow y\end{matrix}$