p. 202

(#1a) $\begin{cases} x' = -2x - 3y \\ y' = -x + 4y \end{cases}$

$\Big\downarrow \; \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$\vec{x}' = \begin{pmatrix} -2 & -3 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} -2 & -3 \\ -1 & 4 \end{pmatrix}}_{\text{coeff matrix}} \vec{x}$$

#1d) $\begin{cases} x' = -2x - y \\ y' = 0x - 4y \end{cases}$

careful! make sure variables line up

$$\vec{x}' = \underbrace{\begin{pmatrix} -2 & -1 \\ 0 & -4 \end{pmatrix}}_{\text{coeff matrix}} \vec{x}$$

(#1c) $\begin{cases} x' = -2x + 0y \\ y' = x + 0y \end{cases} \rightsquigarrow \vec{x}' = \underbrace{\begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix}}_{\text{coeff matrix}} \vec{x}$

technically there!!

(#2)

fr (#1a)

Compute $\det\begin{pmatrix} -2 & -3 \\ -1 & 4 \end{pmatrix} = (-8) - 3 = -11 \neq 0$

So, the equilibrium is $x^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

soln to

$$\begin{pmatrix} -2 & -3 \\ -1 & 4 \end{pmatrix}\vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\vec{x}'$

for #1d $\det\begin{pmatrix} -2 & -1 \\ 0 & -4 \end{pmatrix} = 8 - 0 = 8 \neq 0$

So $x^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

for #1c $\det\begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix} = 0 - 0 = 0$

So there will be ∞-many equilibria!

Multiply out

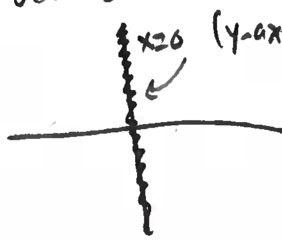
$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$\vec{x}'$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2x \\ x \end{pmatrix} \Rightarrow \begin{cases} 0 = -2x \\ 0 = x \end{cases}$$

same eqt (multiples)

So, "$x=0$" defines a line of equilibria



#3b)

$$\begin{cases} x' = \boxed{-x+3y} -6 \\ y' = \boxed{x+2y} -1 \end{cases}$$

$$(\star) \qquad \vec{x}\,' = \underbrace{\begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}} + \underbrace{\begin{pmatrix} -6 \\ -1 \end{pmatrix}}$$

$$\det\begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix} = -2-3 = -5 \neq 0 \Rightarrow \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix} \text{ invertible}$$

and $$\begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{\det\begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}}\begin{pmatrix} 2 & -3 \\ -1 & -1 \end{pmatrix}$$

$$= -\frac{1}{5}\begin{pmatrix} 2 & -3 \\ -1 & -1 \end{pmatrix}$$

Equilibria: $\vec{x}\,' = 0$, so $(\star)$ becomes

$$\vec{0} = \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -6 \\ -1 \end{pmatrix}$$

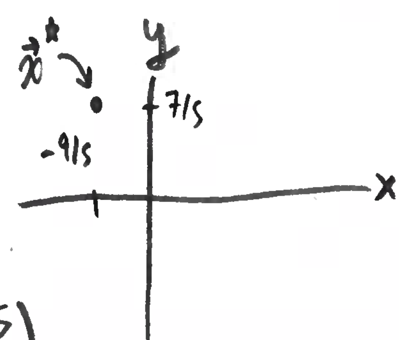
$$\begin{pmatrix} 6 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$\downarrow$ mult. on left by $\begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}^{-1}$

$$\begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 6 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$-\frac{1}{5}\begin{pmatrix} 2 & -3 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 6 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$-\frac{1}{5}\begin{pmatrix} 9 \\ -7 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \vec{x}^{\,*} = \begin{pmatrix} -9/5 \\ 7/5 \end{pmatrix}$$



§4.3 Eigenvalue Problem

Matrix method to solve $\vec{x}\,' = A\vec{x}$

now this is **nonzero**

inspiration from earlier in class:

GUESS $\vec{x} = \vec{v}\,e^{\lambda t}$

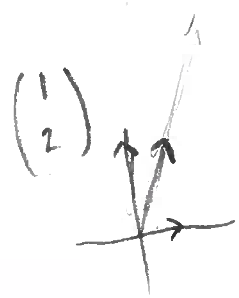
λ is constant

unknowns

$\vec{v}$ is a nonzero constant

From $\vec{x} = \vec{v}e^{\lambda t}$, we get $\vec{x}\,' = \lambda \vec{v} e^{\lambda t}$

Now substitute into DE $\vec{x}\,' = A\vec{x}$:



$$\lambda \vec{v} e^{\lambda t} = A\vec{v}e^{\lambda t}$$

$\downarrow$ divide by $e^{\lambda t}$ (can't div by $\vec{v}$ b/c it is a vector)

$$\boxed{A\vec{v} = \lambda \vec{v}}$$

"algebraic eigenvalue problem"

<u>Remark 4.26</u>:

We say "$\vec{v}$ is an eigenvector of A" if $A\vec{v} = \lambda\vec{v}$ for some constant λ. The constant λ is called "eigenvalue of A".

The pair $(\vec{v}, \lambda)$ called an eigenpair.

All eigenpairs give a soln' $\vec{x} = \vec{v}e^{\lambda t}$ of $\vec{x}\,' = A\vec{x}$.

Solving eigenvalue problem:

Rewrite

$$A\vec{v} = \lambda\vec{v}$$

as

$$A\vec{v} - \lambda\vec{v} = \vec{0}$$

$$\underbrace{(A - \lambda I)}_{\text{a matrix}}\vec{v} = \vec{0}$$

Need ∞-many solns, so by Thm from 24 March notes p.5,

think about things like $x(t) = c_1e^t + c_2e^{-t}$ kind of stuff

We need

$$\det(A - \lambda I) = 0$$

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\lambda I = \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$,

so

$$\begin{aligned}
\det(A - \lambda I) &= \det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) \\
&= \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \\
&= (a-\lambda)(d-\lambda) - bc \\
&= ad - a\lambda - \lambda d + \lambda^2 - bc \\
&= \lambda^2 - (a+d)\lambda + (ad-bc) \\
&= \lambda^2 - \mathrm{tr}(A)\lambda + \det(A)
\end{aligned}$$

Sum of the main diagonal

$$\mathrm{tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a+d$$