Ex: $\begin{pmatrix} 4 & 1 \\ 8 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\det\begin{pmatrix} 4 & 1 \\ 8 & 2 \end{pmatrix} = 4(2)-(1)(8) = 8-8=0 \Rightarrow$ expect probably ∞-many soln

Write as system:

$4x_1 + x_2 = 0 \quad$ (i)

$8x_1 + 2x_2 = 0 \quad$ (ii)

$\frac{1}{2}x = 5$

$x = 10$

$\boxed{\text{Notice (ii)} = 2\text{(i)}}$ → Significance is that no new information is encoded in 2^{nd} equation.

So only one is useful:

$4x_1 + x_2 = 0$ ← tells us relationship b/w variables

$\boxed{x_2 = -4x_1}$

$\Rightarrow$ Get ∞-many solns by picking x_1 & using $x_2 = -4x_1$:

$x_1 = 2 \longrightarrow x_2 = -8 \rightsquigarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -8 \end{pmatrix}$ solves system

$x_1 = 5 \longrightarrow x_2 = -20 \rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ -20 \end{pmatrix}$ ———

In general, $\vec{x} = \begin{pmatrix} x_1 \\ -4x_1 \end{pmatrix} = x_1\begin{pmatrix} 1 \\ -4 \end{pmatrix}$

<u>Def</u>: Two vectors $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ are called <u>independent</u> if they are not multiples of each other. Otherwise <u>dependent</u>.

<u>Note</u>: you'll get a robust def in linear algebra

<u>§4.2.2</u> DE's + equilibria

$$\begin{cases} x' = ax + by \\ y' = cx + dy \end{cases} \rightsquigarrow \text{let } \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\Downarrow$$

$$\boxed{\vec{x}\,' = A\vec{x}}$$

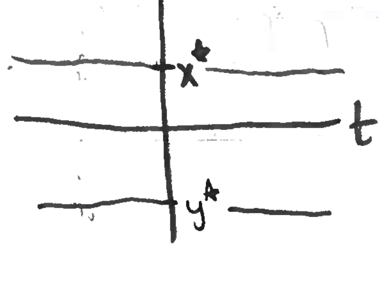
An <u>equilibrium soln</u> of $\vec{x}\,' = A\vec{x}$ is a constant solution $\vec{x}^{\star} = \begin{pmatrix} x^{\star} \\ y^{\star} \end{pmatrix}$ such that $A\vec{x}^{\star} = \vec{0}$.

"set $\vec{x}\,' = 0$"

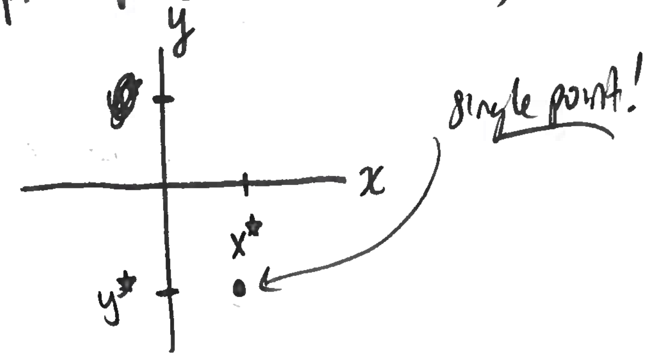
constants

<u>Two interpretations</u>

component plots



phase plane ("parametric plot")



Theorem: The system $\vec{x}' = A\vec{x}$ has

1. unique equilibrium if $\det A \neq 0$
2. straight line of equilibria if $\det A = 0$

Example: Find equilibria of $\vec{x}' = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \vec{x}$

Compute $\det \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} = 2(2) - 4(1) = 0$

System for equilibria is $\vec{0} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$:

multiples
$$2x_1 + x_2 = 0 \quad (i)$$
$$4x_1 + 2x_2 = 0 \quad (ii)$$

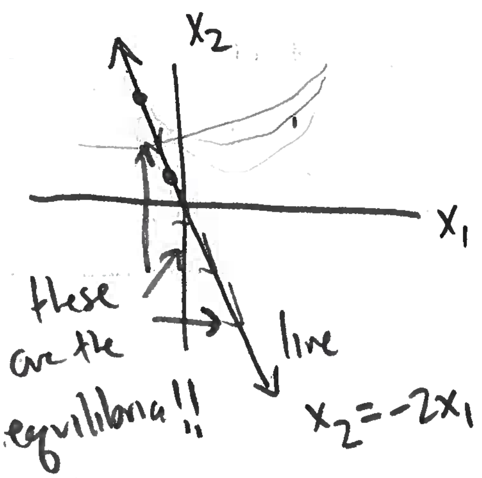
Solve (i) for x_2: $x_2 = -2x_1$

The solutions are

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -2x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

free variable

x_1	$\vec{x}$
0	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
1	$\begin{pmatrix} 1 \\ -2 \end{pmatrix}$
2	$\begin{pmatrix} 2 \\ -4 \end{pmatrix}$
3	$\begin{pmatrix} 3 \\ -6 \end{pmatrix}$
⋮	⋮



The system $A\vec{x}=\vec{0}$ always has $\begin{pmatrix}0\\0\end{pmatrix}$ as a solution. So, $\vec{x}'=A\vec{x}$ always has $\begin{bmatrix}0\\0\end{bmatrix}$ as an equilibrium (since $\vec{x}'=0$)

If $\begin{bmatrix}0\\0\end{bmatrix}$ only crit pt, then we call it <u>isolated</u>

Otherwise <u>nonisolated</u>.

<u>Remark 4.24</u>: System

$$\begin{cases} x' = A\vec{x}+\vec{b} \\ \det(A)\neq 0 \end{cases}$$

is nonhomogeneous.

Crit pt (aka equilibrium) found by solving

$$\vec{0}=A\vec{x}+\vec{b}$$

$$\downarrow$$

$$A\vec{x}=-\vec{b}$$

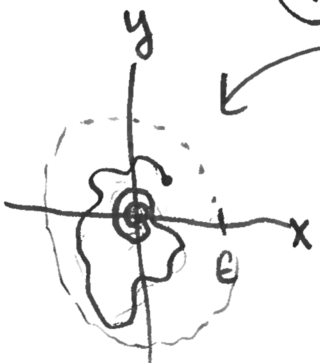
$$\downarrow$$

$$\boxed{\vec{x}=-A^{-1}\vec{b}} \leftarrow \text{isolated equilibrium}$$

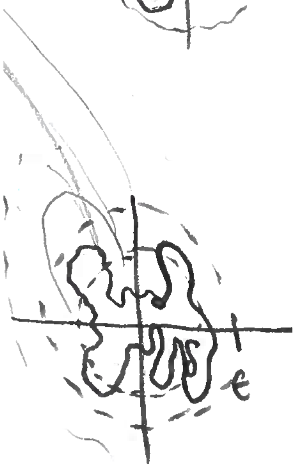
Stability

(5)

(1) isolated equilibrium $x^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is called **asymptotically stable** iff $\exists$ disk $C_\epsilon : x^2 + y^2 < \epsilon^2$ such that all orbits that begin in C_ϵ approach $(0,0)$ as $t \to \infty$



(2) isolated equilib. $x^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ called **stable** if for all disks $C_\epsilon : x^2 + y^2 < \epsilon^2$ there is a corresponding disk $C_\delta : x^2 + y^2 < \delta^2$ inside of C_ϵ s.t. any orbit that starts in C_δ stays in C_ϵ



(3) If not stable, then **unstable**

Note:

* asymptotically stable $\rightarrow$ stable
* asympt. stable $\rightarrow$ origin is an "attractor" or "sink"
* Some unstable are called "repellers" or "source" if they escape all C_ϵ's