§4.2 Matrices + linear systems

matrix ~ grid of numbers

a_{ij} row column

$$A = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{2\ \text{columns}} \Big\}\ 2\ \text{rows}$$

entries

2×2 matrix $\in \mathbb{R}^{2\times 2}$

rows cols

A special kind of matrix is a <u>vector</u>

$$\vec{x} = \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{1\ \text{column}} \Big\}\ 2\ \text{rows}$$

this vector is a 2×1 matrix

$\in \mathbb{R}^{2\times 1}$

Add 2×2 matrices: "add componentwise"

$$A+B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$$

Scalar multiplication: $\alpha \in \mathbb{R}$

$$\alpha A = \alpha \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{pmatrix}$$

Special matrix: zero matrix

$$\mathbf{0} = 0_{2\times 2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Ex: $A=\begin{pmatrix}3 & 1\\ 2 & -1\end{pmatrix}$ $B=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ $\vec{x}=\begin{pmatrix}1\\ 7\end{pmatrix}$ $\vec{y}=\begin{pmatrix}-1\\ 2\end{pmatrix}$

$$A+B=\begin{pmatrix}4 & 3\\ 5 & 3\end{pmatrix},\quad -5B=\begin{pmatrix}-5 & -10\\ -15 & -20\end{pmatrix}$$

$$\begin{aligned}2\vec{x}-3\vec{y}&=2\begin{pmatrix}1\\ 7\end{pmatrix}-3\begin{pmatrix}-1\\ 2\end{pmatrix}\\ &=\begin{pmatrix}2\\ 14\end{pmatrix}-\begin{pmatrix}-3\\ 6\end{pmatrix}\\ &=\begin{pmatrix}5\\ 8\end{pmatrix}\end{aligned}$$

Matrix multiplication:

1st row, 1st col, 2nd col

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}e & f\\ g & h\end{pmatrix}=\begin{pmatrix}ae+bg & af+bh\\ ce+dg & cf+dh\end{pmatrix}$$

2nd row

NOT componentwise!!

$$2\times\cancel{2}\qquad \cancel{2}\times 2\qquad 2\times 2$$

Ex: $A=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$ and $B=\begin{pmatrix}-1 & 2\\ -1 & 3\end{pmatrix}$

$$AB=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}-1 & 2\\ -1 & 3\end{pmatrix}=\begin{pmatrix}-3 & 8\\ -7 & 18\end{pmatrix}$$

$$BA=\begin{pmatrix}-1 & 2\\ -1 & 3\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}=\begin{pmatrix}5 & 6\\ 8 & 10\end{pmatrix}$$

NOT EQUAL!!

in general, $AB\neq BA$

BUT we still have $A(B+C)=AB+AC$

$$A(BC)=(AB)C$$

Special matrix : $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ~ identity matrix

$$AI = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$$

mult by I does not affect

Similarly $IA = A$.

If there is a matrix B such that $AB = I$,
then we call B the inverse of A and we
write $B = A^{-1}$.
"A inverse"

$2 \cdot \frac{1}{2} = 1$
$\frac{1}{2}$ the mult. inverse of 2

In fact,

$$AA^{-1} = I = A^{-1}A.$$

If A^{-1} exists, then we call A invertible (or "nonsingular")
Otherwise we call A singular.

$2 \cdot 2^{-1} = 1$
$2^{-1} \cdot 2 = 1$ ✓

NEVER WRITE "$1/A$" ~ only ever write A^{-1}

The determinant of a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is
a number

$$\det(A) = \det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = ad - bc$$

Turns out if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$ ← Consequence is that A^{-1} exists iff $\det A \neq 0$

Ex: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$\det A = 4 - 6 = -2$, so A is invertible and

$$A^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$$

Is it really A^{-1}?

$$AA^{-1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \checkmark = I$$

Multiply matrix + vector
(2x2) (2x1)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} ae+bf \\ ce+df \end{pmatrix}$$
$2\times\boxed{2 \quad 2}\times 1$ ✓ $\qquad 2\times 1$

note: $A\vec{x}$ is ok ✓

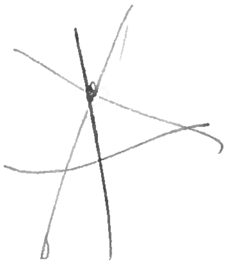
$\vec{x}A$ is not defined!

$2\times 1 \quad 2\times 2$

NOT MATCH!

$$\begin{cases} 2x_1 + 3x_2 = 5 \\ x_1 + x_2 = 6 \end{cases}$$

Look at System

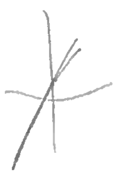


$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

a_{ij} are constants

b_1, b_2 are constant

$x_1, x_2 \sim$ variables



Write this system using matrices:

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{\substack{\text{coefficient} \\ \text{matrix} \\ A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\substack{\text{vector} \\ \text{w/ unknowns} \\ \vec{x}}} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$A \quad \vec{x} \quad = \quad \vec{b}$$

$2x = 3$

$x = \frac{3}{2}$

$$\boxed{A\vec{x} = \vec{b}} \xrightarrow{\text{if } A^{-1} \text{ exists}} A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

<u>Theorem:</u>

(i) if $\det A \neq 0$ then $A\vec{x} = \vec{b}$ has unique soln $\vec{x} = A^{-1}\vec{b}$

and $A\vec{x} = \vec{0}$ —— $\vec{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(ii) if $\det A = 0$, then $A\vec{x} = \vec{0}$ has ∞-many solns

and $A\vec{x} = \vec{b}$ has either no soln or ∞-many solns