§3.2

" Laplace transform converts derivative operations in time domain to multiplication operations in transformed domain."

<u>Thm 3.11</u>: If $x(t)$ is a function and $X(s) = \mathcal{L}\{x\}(s)$, then

$$\begin{cases} \mathcal{L}\{x'\} = sX(s) - x(0) \\ \mathcal{L}\{x''\} = s^2 X(s) - s x(0) - x'(0) \end{cases}$$

$\int u\,dv = uv - \int v\,du$

Show Compute

$$\mathcal{L}\{x'\} \overset{\text{def}}{=} \int_0^\infty x'(t) e^{-st}\,dt \qquad u = e^{-st} \quad dv = x'(t)\,dt$$

$$du = -s e^{-st}\,dt \quad v = x(t)$$

$$= \underbrace{e^{-st} x(t) \Big|_0^\infty}_{} + \underbrace{\int_0^\infty s e^{-st} x(t)\,dt}_{= s\mathcal{L}\{x\}}$$

$$\overset{(s>0)}{=} (0 - x(0)) + sX(s)$$

The calculation for $\mathcal{L}\{x''\}$ is similar!

Ex: What is $\mathcal{L}\{t^2\}$?

$$x(t)=t^2 \qquad x'(t)=2t$$

$$\mathcal{L}\{2t\} = \mathcal{L}\{x'(t)\} \underset{\substack{\uparrow \\ \text{new} \\ \text{formula}}}{=} s\,X(s) - \overset{0^2=0}{\cancel{x(0)}}$$

$$\| $$

$$2\mathcal{L}\{t\}$$

previous work

$$\| $$

$$\frac{2}{s^2}$$

$$\Rightarrow \quad \frac{2}{s^2} = s\,\overbrace{X(s)}^{=\mathcal{L}\{t^2\}}$$

$$\Rightarrow \quad \boxed{\mathcal{L}\{t^2\} = \frac{2}{s^3}}$$

induction can show $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$

§3.2.1 Solving IVP's

$$\begin{tikzcd}
\text{ODE in } x(t) \arrow[r, "\mathcal{L}"] \arrow[d, dashed] & \text{algebraic eqt in } X(s) \arrow[d, "\text{solve}"] \\
\text{find } x(t) & X(s) \arrow[l, "\mathcal{L}^{-1}"]
\end{tikzcd}$$

Ex 3.14: Solve $\begin{cases} x'' + k^2 x = 0 \\ x(0) = 0,\ x'(0) = 1 \end{cases}$ $\quad \lambda^2 + k^2 = 0$

Take $\mathcal{L}$ of ODE:

$$\left[s^2 X(s) - s\,\underbrace{\cancel{(0)}}_{x(0)} - \underbrace{1}_{x'(0)}\right] + k^2 X(s) = 0$$

Solve for $X(s)$

$$X(s)\left[s^2 + k^2\right] = 1$$

$$X(s) = \frac{1}{s^2 + k^2}$$

$$\downarrow \mathcal{L}^{-1}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+k^2}\right\} = \frac{\sin(kt)}{k}$$

$$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$$

$\downarrow$ div by k

$$\mathcal{L}\left\{\frac{\sin(kt)}{k}\right\} = \frac{1}{s^2+k^2}$$

$\downarrow$ write as $\mathcal{L}^{-1}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+k^2}\right\} = \frac{1}{k}\sin(kt)$$

Ex 3.15 Solve $\begin{cases} x' + 2x = e^{-t} \\ x(0) = 0 \end{cases}$ $\mathcal{L}\{e^{-t}\} =$

Soln: Take $\mathcal{L}$:

$$(sX(s) - \underset{x(0)}{0}) + 2X(s) = \frac{1}{s+1}$$

$$X(s)(s+2) = \frac{1}{s+1}$$

$$X(s) = \frac{1}{(s+1)(s+2)}$$

partial fracs

$$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$0s + 1 = (A)(s+2) + B(s+1)$$

$$= (A+B)s + (2A+B)$$

$$\begin{cases} A + B = 0 \\ 2A + B = 1 \end{cases} \quad \boxed{A = -B}$$

$$-2B + B = 1 \quad \boxed{B = -1} \quad \boxed{A = 1}$$

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s-(-1)} - \frac{1}{s-(-2)}$$

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-(-1))(s-(-2))}\right\}$$

$$= \frac{1}{-1+2}\left(e^{-t} - e^{-2t}\right)$$

$$= e^{-t} - e^{-2t}$$

Ex 3.17 What happens if you have

$$\bar{X}(s) = \frac{1}{s^2+3s+6} \leftarrow \text{has complex roots !!}$$

Complete square : $\boxed{\left(x+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 = x^2+bx}$

$$\underset{\uparrow \atop b}{s^2+3s} = \left(s+\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2$$

So,

$$\bar{X}(s) = \frac{1}{\left(s+\frac{3}{2}\right)^2 \underbrace{- \frac{9}{4} + \frac{24}{4}}_{\frac{15}{4}}} = \frac{1}{\left(s+\frac{3}{2}\right)^2 + \left(\frac{\sqrt{15}}{2}\right)^2}$$

$a = -3/2 \quad k = \frac{\sqrt{15}}{2}$

$s-a$

$s-(-3/2)$

k^2

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2+k^2}\right\} = \frac{1}{k} e^{at} \sin(kt)$$

Therefore,

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s-(-3/2))^2+\left(\frac{\sqrt{15}}{2}\right)^2}\right\}$$

$$= \frac{2}{\sqrt{15}} e^{-3/2 t} \sin\left(\frac{\sqrt{15}}{2} t\right)$$

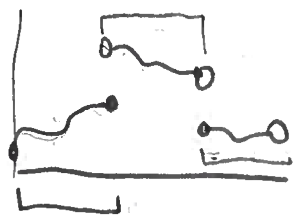
Piecewise continuous forcing fnct

$$\begin{cases} ax''+bx'+cx=f(t) \\ x(0)=x_0,\ x'(0)=x_1 \end{cases}$$

if f ctn $\rightsquigarrow$ could always use var of parameters

if f "nice" $\rightsquigarrow$ could use undet. coeffs

if f piecewise ctn:



var of param → make new IVP w/ new IC's → var of per → new IC's → var of param