# §2.4.2 Variation of Parameters

$$x'' + p(t)x' + q(t)x = f(t)$$

Method: solve

$$x'' + p(t)x' + q(t)x = 0$$

$$\downarrow$$

$$x_h(t) = c_1 x_1(t) + c_2 x_2(t)$$

Make guess of

$$x_p(t) = d_1(t)\,x_1(t) + d_2(t)\,x_2(t)$$

unknown

Known ~ in $x_h$

eventually,

$$d_1(t) = -\int \frac{x_2(t)f(t)}{W(t)}\,dt \quad \text{and} \quad d_2(t) = \int \frac{x_1(t)f(t)}{W(t)}\,dt$$

where

$$\underbrace{W(t) = x_1(t)x_2'(t) - x_1'(t)x_2(t)}_{\text{"Wronskian"}} \qquad \left( = \det\begin{bmatrix} x_1 & x^2 \\ x_1' & x_2' \end{bmatrix} \right)$$

$\boxed{\sin^2 t + \cos^2 t = 1}$

<u>Ex</u>: $x'' + 9x = 3\sec(3t)$ ← $f(t)$

$x_1' = -3\sin(3t)$

$x_2' = 3\cos(3t)$

Solve $x'' + 9x = 0 \longrightarrow \lambda^2 + 9 = 0 \longrightarrow \lambda = \pm\sqrt{-9} = \pm 3i$

$$x_h(t) = c_1 \underbrace{\cos(3t)}_{x_1(t)} + c_2 \underbrace{\sin(3t)}_{x_2(t)}$$

$u = \cos(3t)$

$-\frac{1}{3}du = \sin(3t)dt$

$$d_1(t) = -\int \frac{\sin(3t)\cdot 3\sec(3t)}{\cancel{3}\cos^2(3t) - (-\cancel{3}\sin^2(3t))}\,dt = -\int \frac{\sin(3t)}{\cos(3t)}\,dt = \frac{1}{3}\int \frac{1}{u}\,du$$

$$= \frac{1}{3}\ln(u) = \frac{1}{3}\ln(\cos(3t))$$

$$d_2(t) = \int \frac{\cos(3t) \cdot \cancel{3}\sec(3t)}{\cancel{3}}\,dt = \int 1\,dt = t$$

$$x_p(t) = d_1(t)x_1(t) + d_2(t)x_2(t)$$
$$= \frac{1}{3}\ln(\cos(3t))\cos(3t) + t\sin(3t)$$

$$x(t) = x_h(t) + x_p(t)$$

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#1b p.124-125 $\quad x'' - x = te^t$

$x_2' = -e^{-t}$

$x_1' = e^t$

homogen: $\lambda^2 - 1 = 0 \rightarrow \lambda = \pm 1$

$$\rightarrow x_h(t) = c_1 e^t + c_2 e^{-t}$$
$x_1 \qquad x_2$

$\int u\,dv = uv - \int v\,du$

$$d_1(t) = -\int \frac{e^{-t}\,te^t}{-1-1}\,dt = \frac{1}{2}\int t\,dt = \frac{1}{2}\frac{t^2}{2} = \frac{t^2}{4}$$

wronskian

$(uv)' = u'v + uv'$

$\downarrow \int$

$uv = \int vu' + \int uv'$

$uv = \int v\,du + \int u\,dv$

$$d_2(t) = \int \frac{e^t \cdot te^t}{-2}\,dt = -\frac{1}{2}\int te^{2t}\,dt$$

$$= -\frac{1}{2}\left[\frac{t}{2}e^{2t} - \int \frac{1}{2}e^{2t}\,dt\right] \qquad u = t \quad dv = e^{2t}dt$$

$\downarrow \qquad \downarrow$

$du = dt \qquad v = \frac{1}{2}e^{2t}$

$$= -\frac{t}{4}e^{2t} + \frac{1}{4}\cdot\frac{1}{2}e^{2t}$$
$$= -\frac{t}{4}e^{2t} + \frac{1}{8}e^{2t}$$

So, $x_p(t) = \frac{t^2}{4}e^t + \left(\frac{1}{8} - \frac{t}{4}\right)e^{2t}e^{-t} = \left(\frac{t^2}{4} - \frac{t}{4} + \frac{1}{8}\right)e^t$

Ex: $x'' + x = \frac{1}{t+1}$

homogen: $\lambda^2 + 1 = 0 \rightarrow \lambda = \pm i$

$x_1' = -\sin(t)$

$x_2' = \cos t$

$x_h(t) = c_1 \cos(t) + c_2 \sin(t)$

$x_1 \qquad x_2$

$$d_1(t) = -\int_0^t \frac{\sin(\tau) \cdot \frac{1}{\tau+1}}{\cos^2(\tau) + \sin^2\tau} \, d\tau = -\int_0^t \frac{\sin(\tau)}{\tau+1} \, d\tau$$

can't further simplify

$$d_2(t) = \int_0^t \frac{\cos(\tau) \cdot \frac{1}{\tau+1}}{1} \, d\tau$$

$$x_p(t) = -\cos(t) \int_0^t \frac{\sin(\xi)}{\xi+1} \, d\xi + \sin(t) \int_0^t \frac{\cos(\xi)}{\xi+1} \, d\xi$$

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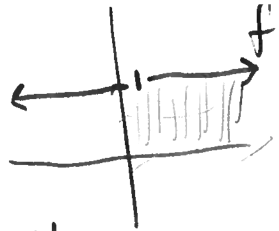
## §3.1 Laplace transform

The Laplace transform of fnct $f$ is the function $\mathcal{L}\{f\}$ given by

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t) e^{-st} \, dt$$

$$= \lim_{b \to \infty} \int_0^b f(t) e^{-st} \, dt$$

**Ex**: Let $f(t)=1$



$s$ SSSS

$$\mathcal{L}\{f\}(s) \overset{def}{=} \int_0^\infty 1\cdot e^{-st}\,dt$$

$u=-st$

$-\frac{1}{s}du = dt$

$t=0 \to u = -s\cdot 0 = 0$

$t=b \to u = -sb$

$$= \lim_{b\to\infty} \int_0^b e^{-st}\,dt$$

$$= -\frac{1}{s}\lim_{b\to\infty} \int_0^{-sb} e^{u}\,du$$

$$\lim_{b\to\infty} e^{-sb} = \begin{cases} 0, & s>0 \\ \underline{\underline{NO}} & s=0 \\ \infty, & s<0 \end{cases}$$

$(Re(s)>0)$

$$= -\frac{1}{s}\lim_{b\to\infty}\left[\overset{0}{e^{-sb}} - 1\right]$$

$(s>0)$

$$= -\frac{1}{s}[0-1] = \frac{1}{s}$$

So, $\boxed{\mathcal{L}\{f\}(s) = \frac{1}{s}}$

$$\boxed{1 \overset{\mathcal{L}}{\longmapsto} \frac{1}{s}}$$