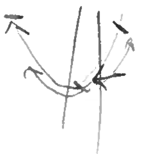
Resonance

Phenomenon that occurs when forcing function's frequency matches that of the fundamental soln of homogeneous equation.



Ex 2.26 LC circuit ~ $L=1$, $\omega_0^2 = \frac{1}{C}$

$$\begin{cases} Q'' + \omega_0^2 Q = \sin(\omega t) \\ Q(0)=0,\ Q'(0)=1 \end{cases}$$

ω is not a specific freq ($\omega \neq 0$)

solve homogen

Step 1: $Q'' + \omega_0^2 Q = 0$

$\lambda^2 + \omega_0^2 = 0 \rightarrow \lambda = \pm \omega_0 i$

>0

$\Rightarrow Q_h(t) = c_0 \cos(\omega_0 t) + c_1 \sin(\omega_0 t)$

Solve nonhomog.

Step 2: $Q_p(t) = A\sin(\omega t)$

$Q_p' = A\omega\cos(\omega t)$ $\quad Q_p'' = -A\omega^2 \sin(\omega t)$

Plug in: $-A\omega^2 \sin(\omega t) + \omega_0^2 A \sin(\omega t) = \sin(\omega t)$

$(A\omega_0^2 - A\omega^2 - 1)\sin(\omega t) = 0$

$\Rightarrow A\omega_0^2 - A\omega^2 - 1 = 0$

$$\boxed{A = \frac{1}{\omega_0^2 - \omega^2}}$$

So we have general solution

$$Q(t) = Q_h(t) + Q_p(t)$$

$$= c_0 \cos(w_0 t) + c_1 \sin(w_0 t) + \frac{1}{w_0^2 - w^2} \sin(wt)$$

<u>Step 3</u>: Find c_0 and c_1.

$$Q'(t) = -c_0 w_0 \sin(w_0 t) + c_1 w_0 \cos(w_0 t) + \frac{w}{w_0^2 - w^2} \cos(wt)$$

$$\underbrace{0}_{\text{given}} = Q(0) = \underbrace{c_0(1) + \cancel{c_1(0)} + \cancel{\frac{1}{w_0^2 - w^2}(0)}}_{\text{computed}}$$

$$\Rightarrow \boxed{c_0 = 0}$$

$$\underbrace{1}_{\text{given}} = Q'(0) = -c_0 w_0(0) + c_1 w_0(1) + \frac{w}{w_0^2 - w^2}(1)$$

$$1 = c_1 w_0 + \frac{w}{w_0^2 - w^2} \Rightarrow \boxed{c_1 = \frac{1}{w_0} - \frac{(w/w_0)}{w_0^2 - w^2}}$$

So soln is

$$Q(t) = \underbrace{\left(\frac{1}{w_0} - \frac{1}{w_0^2 - w^2}\right)}_{w_0^2 - w^2 - w_0} \sin(wt) + \frac{1}{w_0^2 - w^2} \sin(wt)$$

Ex 2.28 $Q'' + 2\sigma Q' + 2Q = \sin(\sqrt{2}t)$ $\quad \sigma > 0$, $\boxed{\text{SMALL}}$

$$\frac{\sqrt{x}}{2} = \sqrt{\frac{x}{4}}$$

Step 1: $\lambda^2 + 2\sigma\lambda + 2 = 0$

$$\lambda = \frac{-2\sigma \pm \sqrt{4\sigma^2 - 4(2)}}{2}$$

$$= \frac{-2\sigma \pm \sqrt{4\sigma^2 - 8}}{2} = -\sigma \pm \sqrt{\sigma^2 - 2}$$

σ small enough so $\sigma^2 - 2 < 0$

$2 - \sigma^2 > 0$

$$\Downarrow$$

$$Q_h(t) = c_1 e^{-\sigma t}\cos\left(\sqrt{2-\sigma^2}\,t\right) + c_2 e^{-\sigma t}\sin\left(\sqrt{2-\sigma^2}\,t\right)$$

Step 2: $Q_p(t) = A\sin(\sqrt{2}t) + B\cos(\sqrt{2}t)$

$Q_p' = A\sqrt{2}\cos(\sqrt{2}t) - B\sqrt{2}\sin(\sqrt{2}t)$

$Q_p'' = -2A\sin(\sqrt{2}t) - 2B\cos(\sqrt{2}t)$

$\sqrt{\sigma^2 - 2} = \sqrt{2-\sigma^2}\, i$

$\sqrt{-4} = \sqrt{4}\, i$

$$\left[-2A\,s(\sqrt{2}t) - 2B\,c(\sqrt{2}t)\right] + 2\sigma\left[A\sqrt{2}\cos(\sqrt{2}t) - B\sqrt{2}\sin(\sqrt{2}t)\right]$$
$$+ 2\left[A\,s(\sqrt{2}t) + B\,c(\sqrt{2}t)\right] = s(\sqrt{2}t)$$

$$\left[-\cancel{2A} - 2\sqrt{2}\sigma B + \cancel{2A}\right]\sin(\sqrt{2}t) + \left[-\cancel{2B} + 2\sqrt{2}A\sigma + \cancel{2B}\right]\cos(\sqrt{2}t) = 1s(\sqrt{2}t) + 0c(\sqrt{2}t)$$

$$\Rightarrow \begin{cases} -2\sqrt{2}\sigma B = 1 \Rightarrow B = \dfrac{-1}{2\sqrt{2}\sigma} \\ 2\sqrt{2}A\sigma = 0 \rightarrow A = 0 \end{cases}$$

$$\Rightarrow Q(t) = c_1 e^{-\sigma t}\cos\left(\sqrt{2-\sigma^2}\,t\right) + c_2 e^{-\sigma t}\sin\left(\sqrt{2-\sigma^2}\,t\right) - \frac{1}{2\sqrt{2}\sigma}\cos(\sqrt{2}t)$$

2^{nd} order linear with variable coeffs

$$x'' + p(t)x' + q(t)x = f(t)$$

Generally → <u>HARD</u> to solve

<u>Airy equation:</u>

$$x'' - tx = 0$$

can't solve in terms of exponentials + trig functs