1

**Thm 2.8:** If $x(t) = g(t) + ih(t)$ solves $ax'' + bx' + cx = 0$, then $g(t)$ and $h(t)$ solve it as well

**Ex 2.9:** Solve $x'' + 2x' + 5x = 0$

**Soln:** Guess: $x(t) = e^{\lambda t}$

$$\downarrow$$

$$\lambda^2 + 2\lambda + 5 = 0 \qquad \sqrt{4-20} = \sqrt{-16} = 4i$$

$$\downarrow$$

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} = \overset{\alpha}{-1} \pm \overset{\beta}{2}i$$

$$\Rightarrow \quad x(t) = e^{(\alpha + \beta i)t} = e^{-t}\left[c_1 \cos(2t) + c_2 \sin(2t)\right]$$

$$= c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t)$$

**Ex 2.10:** Solve $\begin{cases} x'' + 7x = 0 \\ x(0) = 1, \ x'(0) = 2 \end{cases}$

**Soln:** Guess: $x(t) = e^{\lambda t}$

$$e^{\sqrt{7}ti} = \cos(\sqrt{7}t) + i\sin(\sqrt{7}t)$$

$$\lambda^2 + 7 = 0 \rightarrow \lambda^2 = -7$$

$$\lambda = \pm\sqrt{-7} = \pm\sqrt{7}\, i$$

$\Rightarrow$ General Soln is

$$x(t) = c_1 \cos(\sqrt{7}t) + c_2 \sin(\sqrt{7}t)$$

$$x'(t) = -\sqrt{7}c_1 \sin(\sqrt{7}t) + \sqrt{7}c_2 \cos(\sqrt{7}t)$$

IC's

$$1 = x(0) = c_1 \overset{1}{\cancel{\cos}}(0) + c_2 \overset{0}{\cancel{\sin}}(0)$$

$$\boxed{1 = c_1}$$

$$2 = x'(0) = -\sqrt{7}c_1 \overset{0}{\cancel{\sin}}(0) + \sqrt{7}c_2 \overset{1}{\cancel{\cos}}(0)$$

$$2 = \sqrt{7}c_2 \rightarrow \boxed{c_2 = \frac{2}{\sqrt{7}}}$$

Therefore;

$$x(t) = \cos(\sqrt{7}t) + \frac{2}{\sqrt{7}} \sin(\sqrt{7}t)$$

**Summary:**

$\boxed{\text{Solns of } ax''+bx'+cx=0 \text{ w/ e-vals } \lambda_1,\lambda_2}$

| eigenvalues | form of general soln |
|---|---|
| $\lambda_1 \neq \lambda_2$, both real | $x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ |
| $\lambda = \lambda_1 = \lambda_2$ both real | $x(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$ |
| $\lambda_1, \lambda_2$ complex, $\lambda_1 = \alpha+\beta i$, $\lambda_2 = \alpha-\beta i$, $\alpha \neq 0$ | $x(t) = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t)$ |
| $\lambda_1 = \beta i$, $\lambda_2 = -\beta i$ complex $\alpha = 0$ | $x(t) = c_1 \cos(\beta t) + c_2 \sin(\beta t)$ |

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§2.3 $2^{nd}$ order, linear, NONhomogeneous, const coeff

§2.3.1: Undetermined Coeffs

To solve

$$ax'' + bx' + cx = f(t)$$

we first solve homogeneous equation, then "guess" the $x_p(t)$

(Thm 2.14)

## Structure Thm

$$ax'' + bx' + cx = f(t)$$ ← forcing function

has general soln of form   book typo

$$x(t) = \underbrace{c_1 x_1(t) + c_2 x_2(t)}_{\text{soln of homogeneous}} + x_p(t)$$

↖ ANY soln of nonhomogeneous

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Consider this chart for method of undetermined coeffs:

| form of $f(t)$ | form to guess for $x_p(t)$ |
|---|---|
| $\alpha$ | $A$ |
| $\alpha e^{\beta t}$ | $A e^{\beta t}$ |
| polynomial of deg $n$ | $A_n t^n + A_{n-1} t^{n-1} + \cdots + A_1 t + A_0$ |
| $\alpha \sin(wt)$ or $\alpha \cos(wt)$ | $A \cancel{\cos(wt)}\ \sin(wt)$ OR $A \cancel{\sin(wt)}\ \cos(wt)$ |
| $\alpha e^{rt} \sin(wt)$ or $\alpha e^{rt} \cos(wt)$ | $e^{rt}(A\sin(wt) + B\cos(wt))$ |

(Remark 2.15: if guess is same as one of homogen solns, then add a factor of $t$)

Ex 2.17: Find gen soln to

$$x'' + 3x = 4e^{-5t}$$

Soln: Step 1: Solve $x'' + 3x = 0$

$x(t) = e^{\lambda t}$

$\lambda^2 + 3 = 0$

$\lambda = \pm\sqrt{3}\,i \longrightarrow x(t) = c_1 \cos(\sqrt{3}\,t) + c_2 \sin(\sqrt{3}\,t)$

Step 2: Guess $x_p(t) = Ae^{-5t}$ find A

$x_p' = -5Ae^{-5t}$

$x_p'' = 25Ae^{-5t}$

Plug in to $x'' + 3x = 4e^{-5t}$

$$25A\cancel{e^{-5t}} + 3A\cancel{e^{-5t}} = 4\cancel{e^{-5t}}$$

$$25A + 3A = 4$$

$$28A = 4 \longrightarrow A = \frac{4}{28} = \frac{1}{7}$$

Step 3: General soln is

$$x(t) = c_1 \cos(\sqrt{3}\,t) + c_2 \sin(\sqrt{3}\,t) + \frac{1}{7}e^{-5t}$$