Ex 2.6: $x'' + 4x' + 4x = 0$

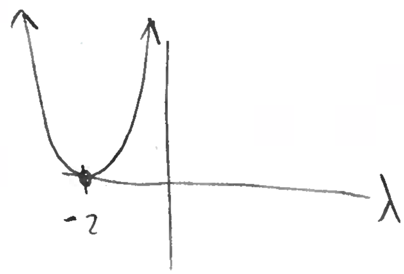
$x = e^{\lambda t} \longrightarrow x' = \lambda e^{\lambda t} \longrightarrow x'' = \lambda^2 e^{\lambda t}$

$\Downarrow$

$\lambda^2 + 4\lambda + 4 = 0$

$(\lambda+2)^2 = 0$

$\lambda = -2$ (double roots)



$\Rightarrow$ Claim that

$x(t) = c_1 e^{-2t} + c_2 t e^{-2t}$ is general soln

def works ?

$x_2(t) = te^{-2t}$

Verify x_2 solves the DE:

$x_2' = e^{-2t} - 2te^{-2t}$

$x_2'' = -2e^{-2t} - 2e^{-2t} + 4te^{-2t}$

$\Downarrow$ plug in ODE

$(\cancel{-4e^{-2t}} + \cancel{4te^{-2t}}) + 4(\cancel{e^{-2t}} - \cancel{2te^{-2t}}) + \cancel{4te^{-2t}}$

$= 0$ ✓

p. 90 #1a w/ IC's from #3

$$\begin{cases} x'' - 4x' + 4x = 0 \\ x(0) = -1, \ x'(0) = 2 \end{cases}$$

Soln: Write $x(t) = e^{\lambda t}$

$$\downarrow$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)^2 = 0$$

$$\lambda = 2$$

$$\Rightarrow x(t) = c_1 e^{2t} + c_2 t e^{2t}$$

$$x'(t) = \underbrace{2c_1 e^{2t} + c_2 e^{2t}} + 2t c_2 e^{2t}$$

$$= (2c_1 + c_2)e^{2t} + 2t c_2 e^{2t}$$

Apply IC's

$$\boxed{-1 = x(0) = c_1} + 0 \qquad c_1 = -1$$

$$2 = x'(0) = 2c_1 + c_2 \longrightarrow 2 = -2 + c_2$$

$$c_2 = 4$$

Particular soln

$$\boxed{x(t) = -e^{2t} + 4te^{2t}}$$

#1b with IC from #3

$$\begin{cases} x'' - 2x' = 0 \\ x(0) = -1, \; x'(0) = 2 \end{cases}$$

<u>Soln</u>: $x(t) = e^{\lambda t}$

$$\downarrow$$

$$\lambda^2 - 2\lambda = 0$$

$$\lambda(\lambda - 2) = 0$$

$$\lambda = 0, 2, \quad 1$$

$$\rightarrow x(t) = c_1 e^{0t} + c_2 e^{2t} \longrightarrow x'(t) = 2c_2 e^{2t}$$

$$= c_1 + c_2 e^{2t}$$

<u>IC's</u>

$$-1 = x(0) = c_1 + c_2 \longrightarrow -1 = c_1 + 1 \rightarrow \boxed{c_1 = -2}$$

$$2 = x'(0) = 2c_2 \longrightarrow \boxed{c_2 = 1}$$

$$\Rightarrow \boxed{x(t) = -2 + e^{2t}}$$

What does "e^{ix}" meen?

$4! = 4 \cdot 3 \cdot 2 \cdot 1$

$i = \sqrt{-1}$

3 facts from calculus 2:

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

$$\begin{tikzcd}
& i^5 = i^9 \arrow[d, equal] & \\
& i \arrow[dr, bend left, "\text{mult by } i"] & \\
i^0 = 1 = i^4 \arrow[ur, bend left, "\text{mult by } i"] & & i^2 = -1 = i^6 \arrow[dl, bend left, "\text{mult by } i"] \\
& i^3 \arrow[ul, bend left, "\text{mult } i"] \arrow[d, equal] & \\
& -i \arrow[d, equal] & \\
& i^7 &
\end{tikzcd}$$

Plug in $i x$ into exponent of e
+ use series to understand it

$$e^{ix} = \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} = \sum_{k=0}^{\infty} \frac{i^k x^k}{k!}$$

$$= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \ldots$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!}$$

no i has i

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)$$

$$e^{ix} = \cos(x) + i\sin(x)$$

$$e^{a+b} = e^a e^b$$

$x = \pi$: $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$

$x = \frac{\pi}{2}$: $e^{i\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$

$x = \frac{\pi}{4}$: $e^{\frac{\pi}{4}i} = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

So.....

$$e^{(\alpha+\beta i)t} = e^{\alpha t + \beta i t} = e^{\alpha t} e^{\beta t i} = e^{\alpha t}\left[\cos(\beta t) + i\sin(\beta t)\right]$$