# §1.4 Linear $1^{st}$ Order ODE's

"normal form" ~ $\boxed{x' + p(t)x \cancel{\phantom{x}} = q(t)}$ ← (1.16)

"forcing function" or "source"

$p, q$ continuous

called <u>homogeneous</u> if $q(t) = 0$

from linear algebra

"linear transformation" $\begin{cases} T : V \to W \\ T(\alpha \vec{v}_1 + \beta \vec{v}_2) = \alpha T(\vec{v}_1) + \beta T(\vec{v}_2) \end{cases}$

property

In this case:

$$\begin{cases} L : \{\text{diff'bl fncts}\} \to \{\text{continuous fncts}\} \\ Lf = f' + p(t) f \end{cases}$$

Here: (1.16) is operator eqt $(Lf)(t) = q(t)$

<u>Notice</u>: $L(\alpha f_1 + \beta f_2) = (\alpha f_1 + \beta f_2)' + p(t)(\alpha f_1 + \beta f_2)$

$$= \alpha f_1' + \beta f_2' + \alpha f_1 p(t) + \beta f_2 p(t)$$

$$= \alpha L f_1 + \beta L f_2'$$

Ex 1.19 :

$\sin(t)\, x' = 3t x + \sqrt{t}$ is linear

* $\sin(t) x' - 3t x = \sqrt{t}$

$$x' + \underbrace{\left(\frac{-3t}{\sin t}\right)}_{p(t)} x = \underbrace{\frac{\sqrt{t}}{\sin(t)}}_{q(t)}$$

source of nonlinearity

* $x' + \cos(t) x x' = 2$ ~ nonlinear

$(x')(1 + \cos(t) x) = 2$ ← can't put into normal form

* $t^2 x' - 4(t+1) = 0$

$$x' = 4 \frac{t+1}{t^2}$$

$$x' + \underbrace{0}_{p(t)=0} x = 4 \underbrace{\frac{t+1}{t^2}}_{q(t)}$$

$$\boxed{x' + p(t)x = q(t)}$$

## How to solve?

1. multiply by $\underbrace{\text{magic factor}}_{}$ integrating factor

$$\mu(t) = e^{\int p(t)\,dt}$$

to get

$$\underbrace{\mu(t)x' + \mu(t)p(t)x}_{} = \mu(t)q(t)$$

NOTICE

$$\frac{d}{dt}\left[x\cdot\mu(t)\right] = x'\mu(t) + x\,\mu'(t)$$

computed

$$= x'\mu(t) + x\left[e^{\int p(t)}\right]\cdot\cancel{\frac{d}{dt}}\left[\cancel{\int} p(t)\right]$$

$$= x'\mu(t) + x\mu(t)p(t)$$

2. $$\frac{d}{dt}\left[x\mu(t)\right] = \mu(t)q(t)$$

3. integrate

$$x\mu(t) = \int \mu(t)q(t)\,dt + C$$

$$\int_{t_0}^{t}\frac{d}{dt}\left[x\mu(s)\right]ds = \int_{t_0}^{t}\mu(s)q(s)\,ds$$

$$x\mu(t) - x(t_0)\mu(t_0) = \int_{t_0}^{t}\mu(s)q(s)\,ds$$

4. Solve for x

$$\boxed{x(t) = \frac{1}{\mu(t)}\underbrace{\int \mu(t)q(t)\,dt}_{} + \frac{C}{\mu(t)}}$$

$$\boxed{x(t) = \frac{1}{\mu(t)}\int_{t_0}^{t}\mu(s)q(s)\,ds + \frac{x(t_0)q(t_0)}{\mu(t)}}$$

Ex 1.21 Solve IVP

$$\begin{cases} x' + \frac{1}{t}x = 1 \\ x(1) = 3 \end{cases}$$

$p(t) = \frac{1}{t}$

$q(t) = 1$

Soln : integrating factor:

$$\mu(t) = e^{\int p(t)\,dt} = e^{\int \frac{1}{t}\,dt} = \cancel{e}^{\ln(t)} = t$$

$(xt)' = x't + x \cdot 1$

multiply DE by $\mu(t)$

$$tx' + \underbrace{t\left(\tfrac{1}{t}\right)}_{=1}x = t$$

$$\frac{d}{dt}\left[tx\right] = t$$

$$tx = \frac{t^2}{2} + C$$

$$\boxed{x(t) = \frac{t}{2} + \frac{C}{t}}$$ ← general soln

Apply initial cond

$$\underset{\substack{\uparrow \\ \text{given}}}{3} = x(1) = \frac{1}{2} + \frac{C}{1} \to C = 3 - \frac{1}{2} = \frac{5}{2} \Rightarrow \boxed{x(t) = \frac{t}{2} + \frac{5}{2} \cdot \frac{1}{t}}$$

particular soln is

**Ex 1.22** $x' + 2x = \sin(t) \leftarrow p(t)=2$

$q(t)=\sin t$

int. factor: $\mu(t) = e^{\int p(t)dt} = e^{2t}$

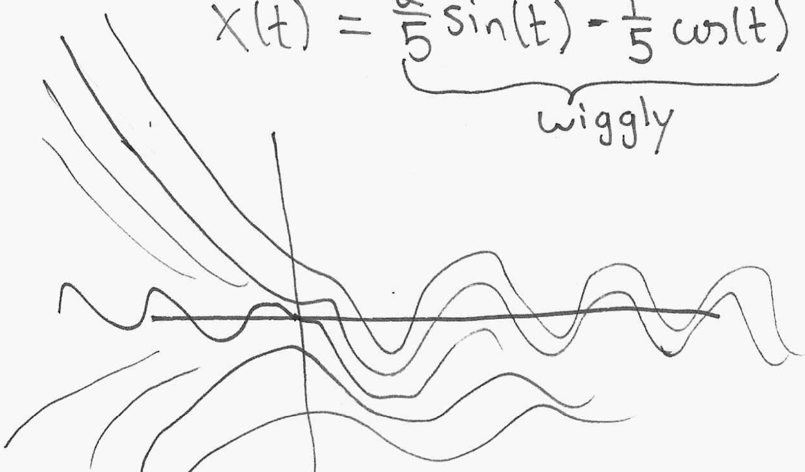
mult by $\mu$: $\underbrace{e^{2t}x' + 2e^{2t}x}_{\frac{d}{dt}\left[e^{2t}x\right]} = e^{2t}\sin(t)$

$$\frac{d}{dt}\left[e^{2t}x\right] = e^{2t}\sin(t)$$

$$e^{2t}x = \int e^{2t}\sin(t)dt$$

$$e^{2t}x = \frac{2}{5}e^{2t}\sin(t) - \frac{1}{5}e^{2t}\cos(t) + C$$

$$x(t) = \underbrace{\frac{2}{5}\sin(t) - \frac{1}{5}\cos(t)}_{\text{wiggly}} + \underbrace{\frac{C}{e^{2t}}}_{\to 0}$$



$(uv)' = u'v + v'u$

$uv = \int u'v + \int v'u$

**Recall**

integration by parts

$$\int u\,dv = uv - \int v\,du$$

$$\int e^{2t}\sin(t)dt = -e^{2t}\cos t + 2\boxed{\int e^{2t}\cos t\,dt}$$

$$\boxed{\begin{array}{ll} u = e^{2t} & dv = \sin t \\ du = 2e^{2t} & v = -\cos(t) \end{array}}$$

Now

$$\int e^{2t}\cos(t)dt = e^{2t}\sin(t) - 2\int e^{2t}\sin t\,dt$$

$u = e^{2t}$ $dv = \cos t$

$du = 2e^{2t}$ $v = \sin t$

$$\Rightarrow \int e^{2t}\sin(t)\,dt = -e^{2t}\cos(t) + 2\left[e^{2t}\sin(t) - 2\int e^{2t}\sin t\,dt\right]$$

$$5\int e^{2t}\sin t\,dt = -e^{2t}\cos(t) + 2e^{2t}\sin t$$

$$\int e^{2t}\sin t\,dt = -\frac{1}{5}e^{2t}\cos(t) + \frac{2}{5}e^{2t}\sin(t)$$