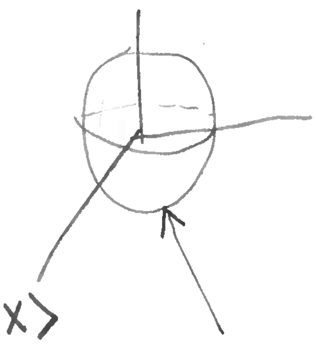
Divergence theorem

Let E be a solid region in $\mathbb{R}^3$ with boundary surface S. Then

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV$$

Ex: Find flux of $\vec{F} = \langle z, y, x \rangle$ over sphere $x^2 + y^2 + z^2 = 1$.



Soln: Compute

$$\begin{aligned} \operatorname{div} \vec{F} &= \nabla \cdot \vec{F} \\ &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle z, y, x \rangle \\ &= \cancelto{0}{\frac{\partial}{\partial x}(z)} + \frac{\partial}{\partial y}(y) + \cancelto{0}{\frac{\partial}{\partial z}(x)} \\ &= \underline{1} \end{aligned}$$

$$E = \left\{ (\rho, \theta, \phi) : \begin{array}{l} 0 \le \rho \le 1 \\ 0 \le \theta \le 2\pi \\ 0 \le \phi \le \pi \end{array} \right\}$$

Compute

$$\iint_S \vec{F} \cdot d\vec{S} \overset{\text{div thm}}{=} \iiint_E 1 \, dV = \int_0^{\pi} \int_0^{2\pi} \int_0^{1} \rho^2 \sin(\phi) \, d\rho \, d\theta \, d\phi$$

$$= \frac{4}{3}\pi$$

Ex: $\vec{F} = \langle xy, y^2 + e^{xz^2}, \sin(xy) \rangle$

where S is boundary surface of region E

bdd by $z = 1 - x^2$ + planes $z=0$, $y=0$, and $y+z=2$.

Compute $\iint_S \vec{F} \cdot d\vec{S}$.

Soln:

Approach as $\iint_S \vec{F} \cdot d\vec{S}$

How to parametrize?

HARD

don't do it

Using div. thm

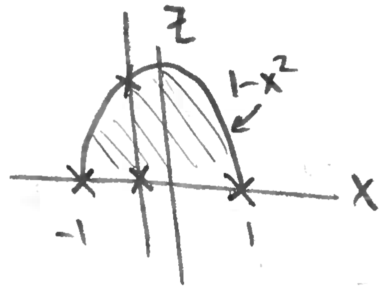
$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}\left(y^2 + e^{xz^2}\right) + \cancel{\frac{\partial}{\partial z}\sin(xy)}^{\,0}$$

$$= y + 2y$$

$$= 3y$$

$1 - x^2 = 0$

$x = \pm 1$



$$\iint_S \vec{F} \cdot d\vec{r} \underset{\text{div thm}}{=} \iiint_E 3y \, dV$$

$$= \iint_D \int_0^{2-z} 3y \, dy \, dA$$

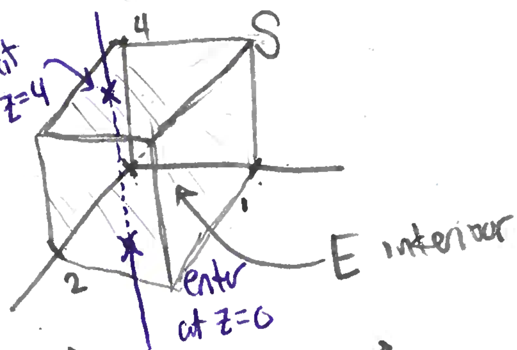
$$= \int_{-1}^{1} \int_0^{1-x^2} \int_0^{2-z} 3y \, dy \, dz \, dx$$

$$= \dots$$

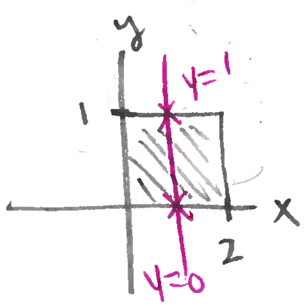
$$= \frac{184}{35}$$

Ex: Compute $\iint_S \vec{F} \cdot d\vec{S}$ where S is surface comprised of the box
$$0 \le x \le 2,\quad 0 \le y \le 1,\quad 0 \le z \le 4$$
and $\vec{F} = \langle \sin(\pi x), zy^3, z^2 + 4x \rangle$

Soln:



$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}$$



$$= \iiint_E \pi\cos(\pi x) + 3zy^2 + 2z \, dV$$

$$= \int_0^2 \int_0^1 \int_0^4 \underbrace{\pi\cos(\pi x) + 3zy^2 + 2z}_{z\pi\cos(\pi x) + \frac{3z^2}{2}y^2 + z^2} \, dz \, dy \, dx$$

$$= \int_0^2 \int_0^1 \underbrace{4\pi\cos(\pi x) + 24y^2 + 16}_{4\pi y\cos(\pi x) + 24\frac{y^3}{3} + 16y} \, dy \, dx$$

$$= \int_0^2 4\pi\cos(\pi x) + \underbrace{8 + 16}_{24} \, dx$$

$$= 4\sin(\pi x) + 24x \Big|_0^2 = 4\sin(2\pi) + 48 = 48$$

General stokes thm

Fund Thm Calc

$$\overbrace{\int_a^b f'(x)\,\partial x}^{1\text{ dim }\int} = f(b)-f(a) = \overbrace{\int_{\{-a,b\}} f}^{\text{integral over 0-dim }\int}$$

FTOLI

$$\overbrace{\int_C \nabla f\cdot d\vec{r}}^{1\text{ dim }\int} = \overbrace{f(\vec{r}(b)) - f(\vec{r}(a))}^{0\text{-dim }\int}$$

Green's

$$\underbrace{\int_C \vec{F}\cdot d\vec{r}}_{1\text{dim }\int} = \underbrace{\iint \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\, dA}_{2\text{dim }\iint}$$

$\langle P, Q\rangle$

Stokes'

$$\underbrace{\int_C \vec{F}\cdot d\vec{r}}_{1\text{dim}} = \underbrace{\iint_S \text{curl } \vec{F}\cdot d\vec{S}}_{2\text{dim}}$$

Div Thm

$$\underbrace{\iint_C \vec{F}\cdot d\vec{S}}_{2\text{dim }\iint} = \underbrace{\iiint_V \text{div } \vec{F}\cdot dV}_{3\text{dim }\iiint}$$