Recall Green's theorem: if C is pos. oriented closed curve surrounding planar region D

$$\oint \langle P, Q \rangle \cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA$$

(could say $C = \partial D$)

Stokes' Theorem Let S be an oriented piecewise smooth surface bounded by piecewise smooth bdy curve C w/ $\oplus$ orientation.

Let $\vec{F}$ be a vector field, then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{F} \cdot d\vec{S}$$

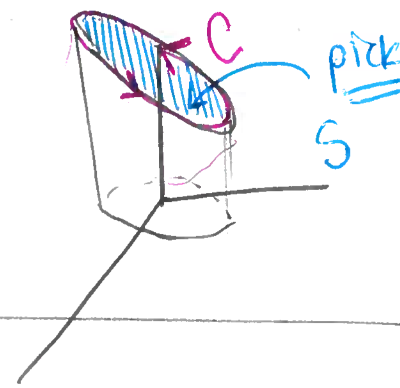
Note: Sometimes right-hand-rule to determine correct orientation

Ex: Use Stokes' thm to evaluate

$$\int_C \vec{F}\cdot d\vec{r} \quad \text{where} \quad \vec{F} = \langle -y^2, x, z^2\rangle$$

and C is curve of intersection of plane $y+z=2$ + cylinder $x^2+y^2=1$.

(orient C as counterclockwise when viewed from above)



Compute $\int_C \vec{F}\cdot d\vec{r}$ directly (w/o Stokes)?

Parametrize C: $\vec{r}(t) = \langle \cos t, \sin t, 2-\sin t\rangle$

$0 \le t \le 2\pi$

$$\vec{r}\,' = \langle -\sin t, \cos t, -\cos t\rangle$$

$$\vec{F}(r) = \langle -\sin^2(t), \cos t, (2-\sin(t))^2\rangle$$

So,

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \sin^3(t) + \cos^2(t) - \cos(t)(2-\sin(t))^2\, dt \overset{\text{computer}}{=} \pi$$

Compute

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{bmatrix}$$

$$= \langle 0, 0, 1+2y\rangle$$

Parametrize S: $\vec{r}(u,v) = \langle u, v, 2-v\rangle$, $(u,v) \in$ 

Calculate

$$r_u = \langle 1, 0, 0 \rangle$$

$$r_v = \langle 0, 1, -1 \rangle$$

$$r_u \times r_v = \langle 0, +1, 1 \rangle$$

Now by Stokes',

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{F} \cdot d\vec{S}$$

polar

$$= \iint_D \langle 0, 0, 1+2v \rangle \cdot \langle 0, 1, 1 \rangle \, dA$$

$u = r\cos\theta$

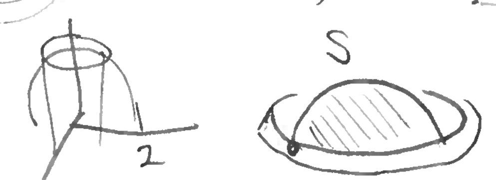
$v = r\sin\theta$

extra

$$= \int_0^{2\pi} \int_0^1 \left(1 + 2r\sin(\theta)\right) r \, dr \, d\theta = \pi$$

Ex: Compute $\iint_S \text{curl}\, \vec{F} \cdot d\vec{S}$ where $\vec{F} = \langle xz, yz, xy \rangle$ and S part of sphere lying inside cylinder $x^2 + y^2 = 1$. $(z \geq 0)$ $x^2 + y^2 + z^2 = 4$

Soln:



Subtract cylinder eqt from sphere eqt:

$$z^2 = 3 \rightarrow z = \pm\sqrt{3}$$

$\Rightarrow$ curve of intersection is $\vec{r}(t) = \langle \cos t, \sin t, \sqrt{3} \rangle$, $0 \leq t \leq 2\pi$

$\vec{r}\,' = \langle -\sin t, \cos t, 0 \rangle$

4

By Stokes',

$$\iint_S \text{curl}\,\vec{F}\, d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_0^{2\pi} \langle \sqrt{3}\cos t, \sqrt{3}\sin t, \cos(t)\sin(t) \rangle \cdot \langle -\sin t, \cos t, 0 \rangle dt$$

$$= \int_0^{2\pi} \underbrace{\sqrt{3}(\cos t)(-\sin(t)) + \sqrt{3}\sin(t)\cos(t)}_{=0} + 0 \, dt$$

$$= \int_0^{2\pi} 0\, dt$$

$$= 0$$