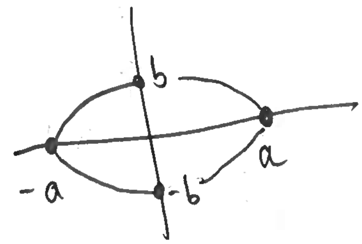
Ex: Find area of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Parametrize ellipse:

$$\begin{cases} \vec{r}(t) = \langle a\cos t,\ b\sin(t) \rangle \\ 0 \le t \le 2\pi \end{cases}$$

From earlier (last class)

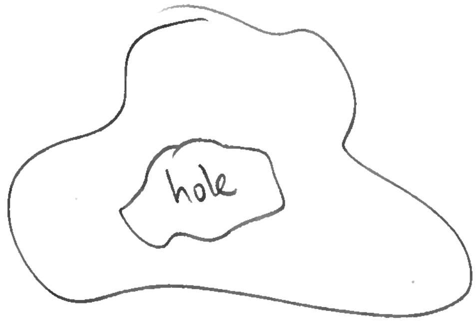
$$\text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx$$

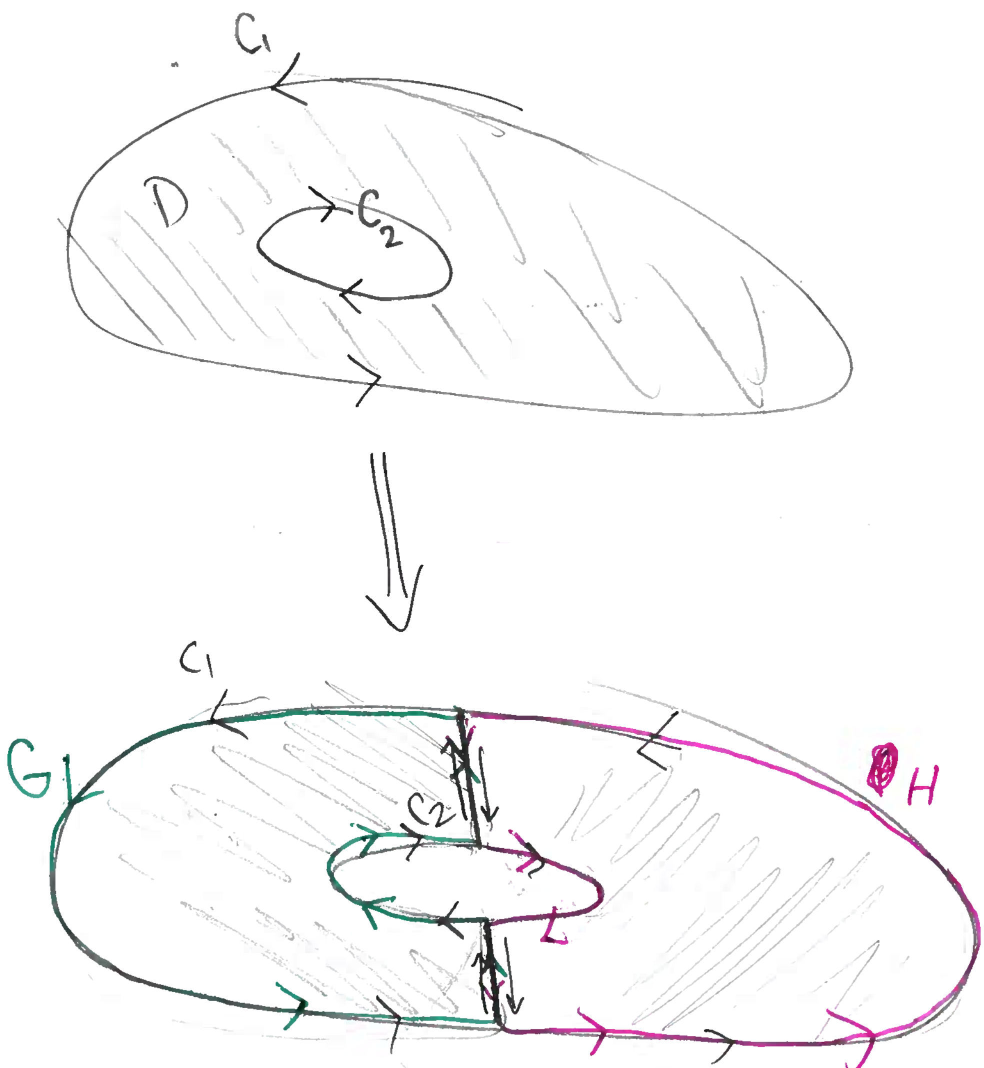
$$= \frac{1}{2}\int_0^{2\pi} a\cos(t)\,b\cos(t) - b\sin(t)(-a\sin(t))\ dt$$

$$= \frac{1}{2}\int_0^{2\pi} ab\underbrace{\left(\cos^2(t) + \sin^2(t)\right)}_{=1} dt$$

$$= \frac{1}{2}ab\int_0^{2\pi} 1\,dt = \frac{2\pi ab}{2} = \pi ab$$

What happens if our region is no longer simply connected?





$$\int_{C_1 \cup C_2} = \int_{G \cup P}$$

So,

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, \partial A = \overbrace{\oint_{C_1 \cup C_2} P\,dx + Q\,dy}^{\text{strange}} = \oint_{G} P\,dx + Q\,dy + \oint_{H} P\,dx + Q\,dy$$

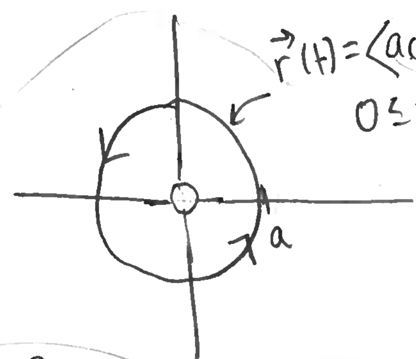
Ex: $\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$

This vector field is not defined at $(x,y) = (0,0)$.

Compute $\oint_C \vec{F}\cdot d\vec{r}$ where C is circle w/ radius a.

$\vec{r}(t) = \langle a\cos t, a\sin t \rangle$

$0 \le t \le 2\pi$



$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left\langle \frac{-a\sin t}{a^2\cos^2(t) + a^2\sin^2(t)}, \frac{a\cos t}{a^2\cos^2(t) + a^2\sin^2 t} \right\rangle \cdot \langle -a\sin t, a\cos t \rangle\, dt$$

$$= \int_0^{2\pi} \frac{a^2\sin^2(t)}{a^2} + \frac{a^2\cos^2(t)}{a^2}\, dt$$

$$= \int_0^{2\pi} 1\, dt = 2\pi$$

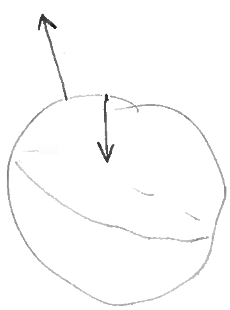
Curl and divergence of vector fields

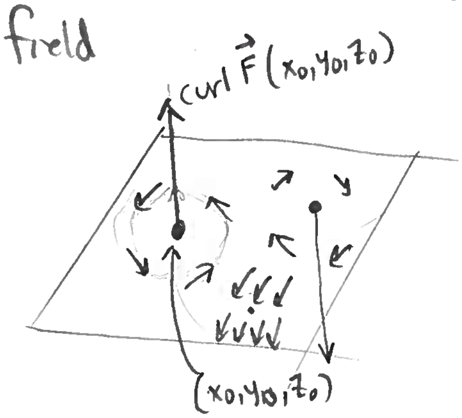
Recall: $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

implies we should think of ∇f as

$\vec{\nabla} f$ where $\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle f$

Curl – measures rotation around points of a vector field





Turns out:

$$\text{curl } \vec{F} \overset{\text{def}}{=} \nabla \times \vec{F} \leftarrow \vec{F} = \langle P, Q, R \rangle$$

$$= \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix}$$

Same thing from Green's theorem

$$= \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right), \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{} \right\rangle$$

Ex: $\vec{F} = \langle xz, xyz, -y^2 \rangle$

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F}$$

$$= \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{bmatrix}$$

$$= \langle -2y - xy, -(-x), yz \rangle$$

$$= \langle -2y - xy, x, yz \rangle$$

Theorem: $\operatorname{curl}(\nabla f) = \vec{0}$ (provided f has continuous partial derivatives)

Proof: $P = \frac{\partial f}{\partial x}$, $Q = \frac{\partial f}{\partial y}$, $R = \frac{\partial f}{\partial z}$

Directly,

$$\operatorname{curl}\nabla f = \langle \underbrace{f_{zy} - f_{yz}}_{=0 \text{ if } f \text{ is nice}}, -\underbrace{(f_{zx} - f_{xz})}_{=0}, \underbrace{f_{yx} - f_{xy}}_{=0} \rangle$$

$$= \langle 0, 0, 0 \rangle$$

Significance: if $\operatorname{curl}\vec{F} \neq 0$, then $\vec{F}$ is not conservative