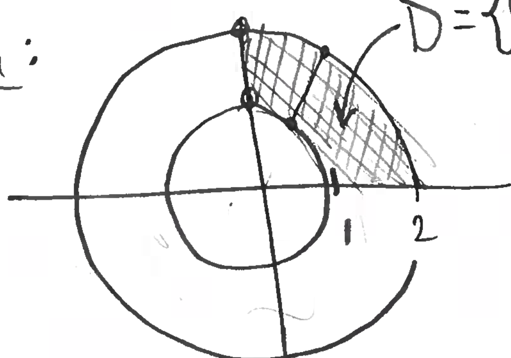
Ex: Calculate

$$\iint_D \frac{x^2}{x^2+y^2}\,dA$$

$x = r\cos\theta$
$y = r\sin\theta$ $\rightarrow$ $x^2+y^2=r^2$

where D is region b/w circles $x^2+y^2=1$ and $x^2+y^2=4$ in quadrant I.

Soln: $D=\left\{(r,\theta):\ \begin{matrix} 1\le r\le 2 \\ 0\le\theta\le\frac{\pi}{2}\end{matrix}\right\}$



So, calculate

extra r

$$\iint_D \frac{x^2}{x^2+y^2}\,dA = \int_0^{\pi/2}\int_1^2 \frac{\cancel{r^2}\cos^2(\theta)}{\cancel{r^2}}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_1^2 r\cos^2(\theta)\,dr\,d\theta$$

Power reduce

$$\cos^2(\theta)=\frac{1+\cos(2\theta)}{2}$$

$$= \int_0^{\pi/2}\left(\frac{4}{2}-\frac{1}{2}\right)\cos^2(\theta)\,d\theta$$

$$= \frac{3}{4}\int_0^{\pi/2} 1+\cos(2\theta)\,d\theta$$

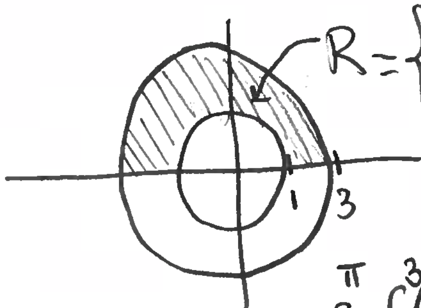
$u = 2\theta$
$\frac{1}{2}du = d\theta$

$$= \frac{3}{4}\left[\theta+\frac{1}{2}\sin(2\theta)\right]_0^{\pi/2} = \frac{3}{4}\left[\left(\frac{\pi}{2}+\frac{1}{2}\overset{0}{\sin(\pi)}\right)-(0+0)\right]$$

$$= \frac{3\pi}{8}$$

Ex: $\iint_R 4x+3y\,dA$ where R is in upper half plane bdd by circles $x^2+y^2=1$, $x^2+y^2=9$.

Soln:



$R=\{(r,\theta): 1\le r\le 3,\ 0\le \theta\le \pi\}$

extra

$$\iint_R 4x+3y\,dA = \int_0^\pi\int_1^3 (4r\cos(\theta)+3r\sin(\theta))\,r\,dr\,d\theta$$

$$= \int_0^\pi\int_1^3 r^2\left[4\cos(\theta)+3\sin(\theta)\right]dr\,d\theta$$

$\frac{r^3}{3}$

$$= \int_0^\pi \left(\frac{3^3}{3}-\frac{1^3}{3}\right)(4\cos\theta+3\sin(\theta))\,d\theta$$

$$= \frac{26}{3}\left[4\sin(\theta)-3\cos(\theta)\right]\Big|_0^\pi$$

$$= \frac{26}{3}\left[(0-3(-1))-(0-3(1))\right]$$

$$= \frac{26}{3}[3+3]=52$$

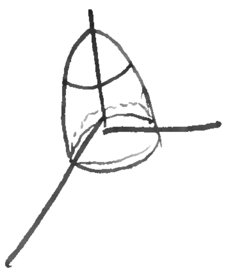
Ex: Find volume of solid bdd by plane $z=0$ & paraboloid $z=1-x^2-y^2$.

xy-plane

$\iint \ldots$?

Soln: What is my region?

Combine $z=0$ and $z=1-x^2-y^2$



$$0=1-x^2-y^2$$

$$\boxed{x^2+y^2=1}$$

$D=\{(r,\theta): 0\le r\le 1,\ 0\le\theta\le 2\pi\}$

$\Rightarrow$ region:

extra

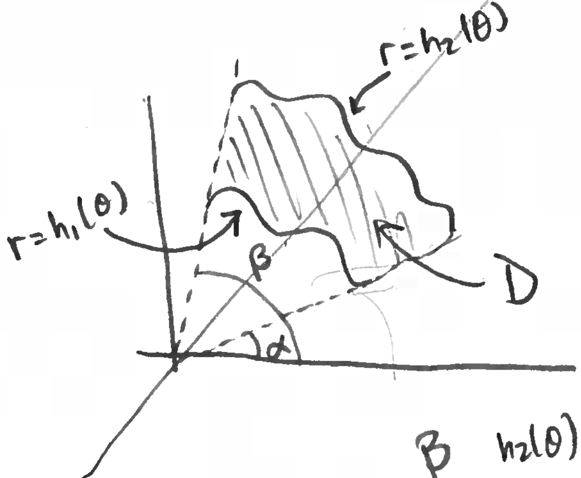
$$\text{Vol}=\iint_D 1-(x^2+y^2)\,dA=\int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta$$

$$=\int_0^{2\pi}\left[\frac{r^2}{2}-\frac{r^3}{3}\right|_0^1 d\theta$$

$\frac{3}{6}-\frac{2}{6}$

$$=\frac{1}{6}\int_0^{2\pi} 1\,d\theta$$

$$=\frac{2\pi}{6}=\frac{\pi}{3}$$

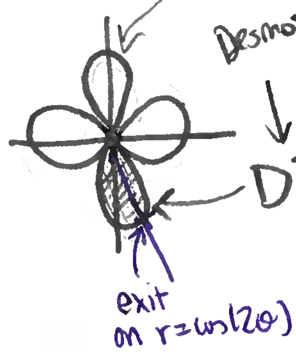


XTRA

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Ex: Use double int to find area enclosed by one loop (leaf) of four-leafed rose $r=\cos(2\theta)$.

Desmos: $\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$



$$D = \left\{ (r,\theta) : \begin{array}{l} 0 < r < \cos(2\theta) \\ \frac{\pi}{4} \le \theta \le \frac{3\pi}{4} \end{array} \right\}$$

extra r

$$\text{Area} = \iint_D 1\, dA = \int_{\pi/4}^{3\pi/4} \int_0^{\cos(2\theta)} r\, dr\, d\theta$$

$$\cos^2(2\theta) = \frac{1+\cos(4\theta)}{2}$$

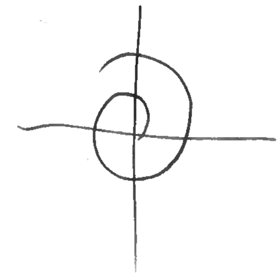
$$= \frac{1}{2}\int_{\pi/4}^{3\pi/4} \cos^2(2\theta)\, d\theta = \frac{1}{4}\int_{\pi/4}^{3\pi/4} 1+\cos(4\theta)\, d\theta$$

$$= \frac{1}{4}\left[\theta + \frac{1}{4}\sin(4\theta)\right]_{\pi/4}^{3\pi/4}$$

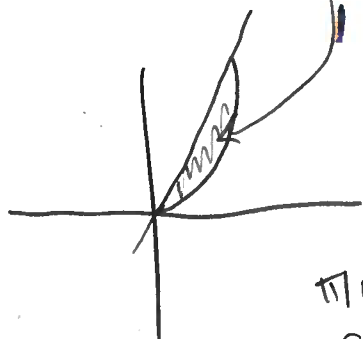
$$= \frac{1}{4}\left[\left(\frac{3\pi}{4} + \frac{1}{4}\cancelto{0}{\sin(3\pi)}\right) - \left(\frac{\pi}{4} + \frac{1}{4}\cancelto{0}{\sin(\pi)}\right)\right] = \frac{\pi}{8}$$

Ex: Sketch D find $\iint_D \frac{x^3+xy^2}{(x^2+y^2)^2}\, dA$

where $D = \left\{(r,\theta) : \begin{matrix} 0<\theta\le \pi/4 \\ 0<r\le 2\theta \end{matrix}\right\}$



Soln:



xtra

$$\iint_D \frac{x^3+xy^2}{(x^2+y^2)^2}\, dA = \int_0^{\pi/4}\int_0^{2\theta} \frac{\cancel{r^3}\cos^3(\theta) + \cancel{r^3}\cos(\theta)\sin^2(\theta)}{\cancel{r^4}\, r^3}\, \cancel{r}\, dr\, d\theta$$

$$= \int_0^{\pi/4}\int_0^{2\theta} \cos(\theta)\underbrace{\left[\cos^2(\theta) + \sin^2(\theta)\right]}_{=1} dr\, d\theta$$

$\int 1\, dr = r + C$

$$= \int_0^{\pi/4} r\cos(\theta)\Big|_{r=0}^{r=2\theta}$$

$(fg)' = f'g + fg'$

$\int fg' = \int (fg)' - \int f'g$

$\int u\, dv = uv - \int v\, du$

$$= 2\int_0^{\pi/4} \theta\cos(\theta)\, d\theta$$

$u = \theta$ $\quad dv = \cos(\theta)$

$du = d\theta$ $\quad v = \sin(\theta)$

$$= 2\left[\theta\sin(\theta)\Big|_0^{\pi/4} - \int_0^{\pi/4} \sin(\theta)\, d\theta\right]$$

antideriv $-\cos(\theta)$

$\sqrt{2}/2$ $\quad \sqrt{2}/2$

$$= 2\left\{\left[\frac{\pi}{4}\sin\left(\frac{\pi}{4}\right) - 0\right] + \left[\cos\left(\frac{\pi}{4}\right) - \cos(0)\right]\right\}$$

$$= \frac{\pi}{2}\frac{\sqrt{2}}{2} + \frac{2\sqrt{2}}{2} - 2 = \frac{\sqrt{2}\pi}{4} + \sqrt{2} - 2$$