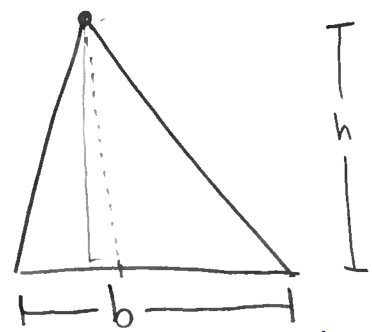
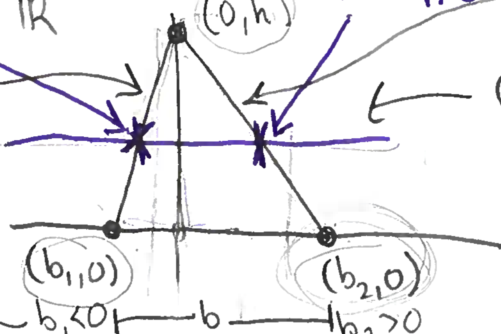
Ex: Recall $\text{Area}(D) = \iint_D 1\, dA$

Goal: find area of a general Δ:



$\text{Area}_\Delta = \frac{1}{2}bh$

embed into $\mathbb{R}^2$



$x = b_1 - \frac{b_1 y}{h}$

$x = -\frac{b_2}{h}y + b_2$

easiest to do as $dxdy$

formula

$\text{slope} = \frac{h-0}{0-b_1} = -\frac{h}{b_1} > 0$

$y - 0 = -\frac{h}{b_1}(x - b_1)$

$\boxed{-\frac{b_1 y}{h} + b_1 = x}$

$b = b_2 - b_1$

formula

$\text{slope} = \frac{h-0}{0-b_2}$

$= -\frac{h}{b_2} < 0$

$y - 0 = -\frac{h}{b_2}(x - b_2)$

$x = -\frac{b_2}{h}y + b_2$

$$\text{Area}(\Delta) = \iint_\Delta 1\, dA = \int_0^h \int_{b_1 - \frac{b_1 y}{h}}^{b_2 - \frac{b_2 y}{h}} 1\, dx\, dy$$

$$= \int_0^h x \Big|_{x = b_1 - \frac{b_1 y}{h}}^{x = b_2 - \frac{b_2 y}{h}} dy$$

$$= \int_0^h \left(b_2 - \frac{b_2 y}{h}\right) - \left(b_1 - \frac{b_1 y}{h}\right) dy$$

$$= \int_0^h \underbrace{(b_2 - b_1)}_{=b} + \underbrace{(b_1 - b_2)}_{=-b} \frac{y}{h}\, dy$$

$$= \int_0^h b - \frac{b}{h} y\, dy = by - \frac{b}{2h}y^2 \Big|_{y=0}^{y=h} = \left(bh - \frac{b}{2h}h^2\right) - (0 + 0)$$

$$= bh - \frac{1}{2}bh$$

$$= \frac{1}{2}bh \ \checkmark$$

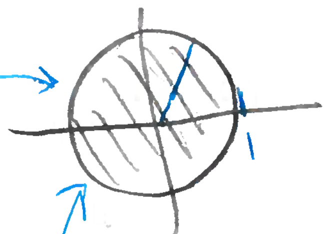
Double integrals in polar coordinates

Suppose we want to calculate

$$\iint_D f(x,y)\,dA$$

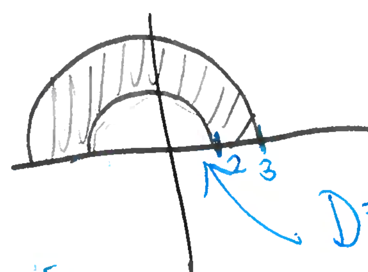
where D looks like

"polar rectangle"



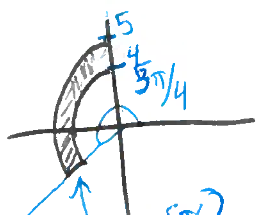
$D=\left\{(r,\theta):\ \begin{array}{l}0\le\theta\le 2\pi\\ 0\le r\le 1\end{array}\right\}$

OR



$D=\left\{(r,\theta):\ \begin{array}{l}0\le\theta\le \pi\\ 2\le r\le 3\end{array}\right\}$

OR



$D=\left\{(r,\theta):\ \begin{array}{l}\pi/2\le\theta\le 5\pi/4\\ 4\le r\le 5\end{array}\right\}$

Recall(?):

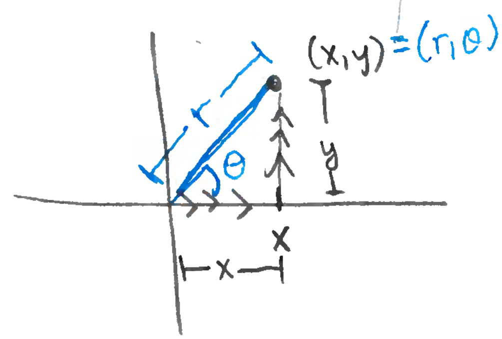
polar coords
$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ x^2+y^2=r^2 \end{cases}$$

$\sin^2\theta+\cos^2\theta=1$

LHS: $r^2(\sim\sim)$

(x,y) ↑ rectangular coords

(r,θ) ↑ polar



TURNS OUT: you can't just replace "dxdy" or "dydx" with "drdθ" or "dθdr"

TURNS OUT:

$$\boxed{dx\,dy = r\,dr\,d\theta}$$

"extra r"

"Jacobian"

Convert to polar

extra r

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta$$

usually angles — usually numbers

Ex: Compute $\iint_D 3x+y\,dA$

where D is disk $x^2+y^2 \le 1$.

$D = \{(r,\theta) : 0 \le r \le 1, \ 0 \le \theta \le 2\pi\}$

Calculate

extra

$$\iint_D 3x+y\,dA = \int_0^{2\pi}\int_0^1 [3r\cos(\theta) + r\sin(\theta)]\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 3r^2\cos(\theta) + r^2\sin(\theta)\, dr\, d\theta$$

$\int r^2 = \frac{r^3}{3}$

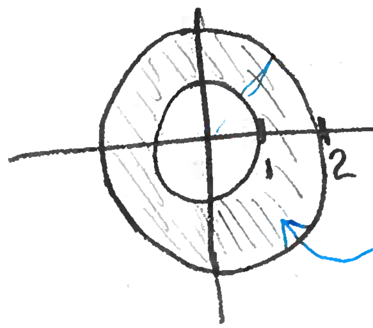
$$= \int_0^{2\pi} \frac{3}{3}\cos(\theta) + \frac{1}{3}\sin(\theta)\, d\theta = \sin(\theta) - \frac{1}{3}\cos(\theta)\Big|_0^{2\pi}$$

$$= \left(0 - \tfrac{1}{3}\right) - \left(0 - \tfrac{1}{3}\right)$$

$$= 0$$

Ex: $\iint_R (x^2+y^2)\, dA$ where D is region between circles $x^2+y^2=1$ and $x^2+y^2=4$.

Soln:

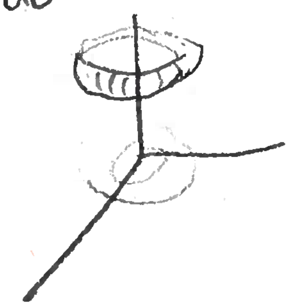


$D = \left\{ (r,\theta) : \begin{array}{l} 1 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{array} \right\}$

So,

extra

$$\iint_R x^2+y^2 \, dA = \int_0^{2\pi}\int_1^2 r^2 \, r \, dr\, d\theta$$



$4 - \frac{1}{4}$

$\frac{16-1}{4}$

$\frac{r^4}{4}$

$$= \int_0^{2\pi}\int_1^2 r^3 \, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{2^4}{4} - \frac{1^4}{4} \, d\theta$$

$$= \frac{15}{4}\int_0^{2\pi} 1 \, d\theta$$

$$= \frac{15}{4}(2\pi) = \frac{15\pi}{2}$$

Calculate $\int_{-\infty}^{\infty} e^{-t^2}\, dt$ ← big deal in stats