Directional derivatives

if $\vec{u} = \langle a, b \rangle$ and $z = f(x,y)$, then

$\|\vec{u}\| = 1$

$$D_{\vec{u}} f(x,y) = a \underbrace{f_x(x,y)} + b \underbrace{f_y(x,y)}$$

Ex: Let $f(x,y) = x^2 + 3x + y^4$

find $D_u f(5,3)$ in direction of $\langle 1, 17 \rangle$.

NOT $\vec{u}$

Soln: Since $\|\langle 1,17 \rangle\| = \sqrt{1+17^2} \neq 1$

we need to normalize it to get $\vec{u}$:

$$\vec{u} = \frac{\langle 1,17 \rangle}{\|\langle 1,17 \rangle\|} = \frac{\langle 1,17 \rangle}{\sqrt{290}} = \left\langle \frac{1}{\sqrt{290}}, \frac{17}{\sqrt{290}} \right\rangle$$

Notice: we could regard $D_u f(x,y)$ as a certain dot product:

$$D_{\vec{u}} f(x,y) = \vec{u} \cdot \underbrace{\langle f_x(x,y), f_y(x,y) \rangle}_{\text{special vector related to } f \text{ — "gradient"}}$$

Formally, if we have $f(x,y)$:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

"del f"
"gradient of f"
"nabla f"

Note: do $\underline{\underline{\text{NOT}}}$ write Δf because it actually means $\nabla \cdot \nabla = \nabla^2$

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$$

$$\begin{aligned} \Delta f &= \nabla \cdot \nabla f \\ &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \cdot \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \end{aligned}$$

the "Laplacian" ~ ties into partial diff'l eqt's

Ex: If $f(x,y) = \sin(x) + e^{xy^2}$

Compute ∇f and $\nabla f(0,1)$.

Soln:

$$\begin{aligned} \nabla f &= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \\ &= \left\langle \cos(x) + y^2 e^{xy^2}, 2xye^{xy^2} \right\rangle \end{aligned}$$

$$\begin{aligned} \nabla f(0,1) &= \left\langle \cos(0) + 1^2 e^0, 0 \right\rangle \\ &= \langle 2, 0 \rangle \end{aligned}$$

Theorem: The maximum value of $D_u f(x,y)$ is $\|\nabla f(x,y)\|$

Morever, that occurs in direction of $\nabla f(x,y)$.

Ex: $f(x,y) = x^2 \ln(y)$ at $P = (1,2)$

a) Find rate of change at $P=(1,2)$ in direction towards $Q = (2,7)$.

$f(1,2) = 1^2 \ln(2) = \ln(2)$

b) In what dir does the max rate of change occur? What is that rate of change?

Soln: $\vec{PQ} = \langle 2,7 \rangle - \langle 1,2 \rangle = \langle 1,5 \rangle$

a) $\vec{u} = \dfrac{\langle 1,5 \rangle}{\|\langle 1,5 \rangle\|} = \dfrac{\langle 1,5 \rangle}{\sqrt{26}} = \left\langle \dfrac{1}{\sqrt{26}}, \dfrac{5}{\sqrt{26}} \right\rangle$

$\nabla f = \left\langle 2x\ln(y), \dfrac{x^2}{y} \right\rangle$

$D_{\vec{u}} f(x,y) = \vec{u} \cdot \nabla f = \dfrac{2}{\sqrt{26}} 2x\ln(y) + \dfrac{5}{\sqrt{26}} \dfrac{x^2}{y}$

$D_{\vec{u}} f(1,2) = \dfrac{4}{\sqrt{26}} \ln(2) + \dfrac{5}{\sqrt{26}} \dfrac{1}{2} > 0$

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b) Max rate of change is in dir of

$\vec{u} = \dfrac{\nabla f(1,2)}{\|\nabla f(1,2)\|}$ $\quad \nabla f(1,2) = \left\langle 2\ln(2), \dfrac{1}{2} \right\rangle$

$\|\nabla f(1,2)\| = \sqrt{4\ln(2)^2 + \frac{1}{4}}$

$D_{\vec{u}} f = \vec{u} \cdot \nabla f(1,2) = \dfrac{1}{\|\nabla f(1,2)\|} \nabla f(1,2) \cdot \nabla f(1,2) = \dfrac{\|\nabla f(1,2)\|^2}{\|\nabla f(1,2)\|}$

$= \|\nabla f(1,2)\|$

Ex: $f(x,y,z) = x^2y + yz^3 + xyz$

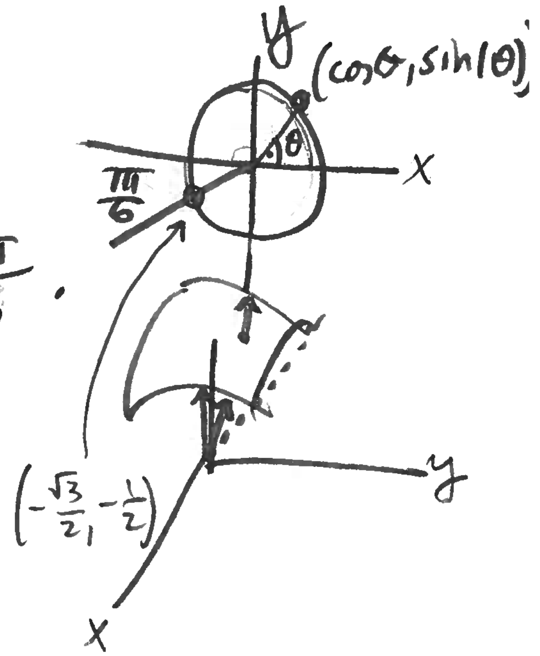
$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$= \left\langle 2xy + yz, x^2 + z^3 + xz, 3yz^2 + xy \right\rangle$$

Ex: Find dir deriv of

$$f(x,y) = \cos(2x - y)$$

at $(2,7)$ in dir of $\theta = \frac{7\pi}{6}$.



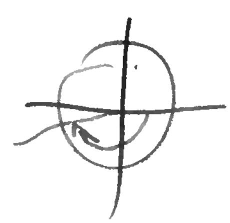
Soln: $\vec{u} = \left\langle \cos\left(\frac{7\pi}{6}\right), \sin\left(\frac{7\pi}{6}\right) \right\rangle$

$$= \left\langle -\frac{\sqrt{3}}{2}, \frac{-1}{2} \right\rangle$$

$$\|\vec{u}\| = \sqrt{\left(\frac{-\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = 1$$

$$\nabla f = \left\langle -\sin(2x - y)(2), -\sin(2x-y)(-1) \right\rangle$$

$$= \left\langle -2\sin(2x-y), \sin(2x-y) \right\rangle$$



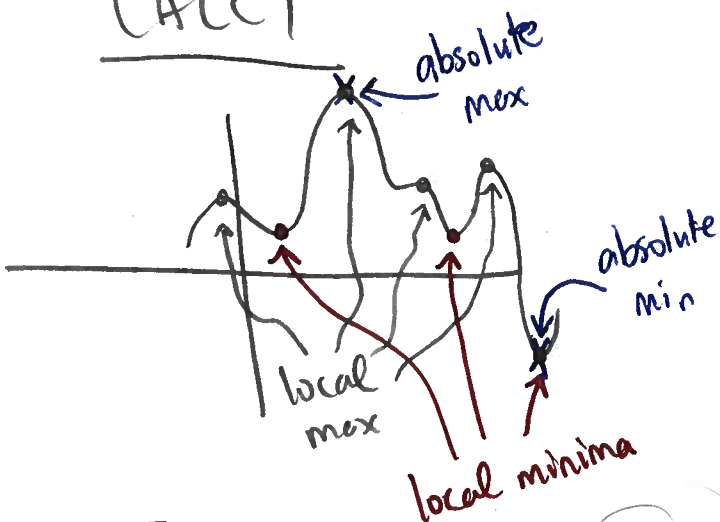
$$D_{\vec{u}} f(2,7) = \vec{u} \cdot \left\langle -2\sin(-3), \sin(-3) \right\rangle$$

$$= -\frac{\sqrt{3}}{2}\left(-2\sin(-3)\right) + \left(-\frac{1}{2}\right)\sin(-3)$$

$$\approx -0.17$$

Extrema

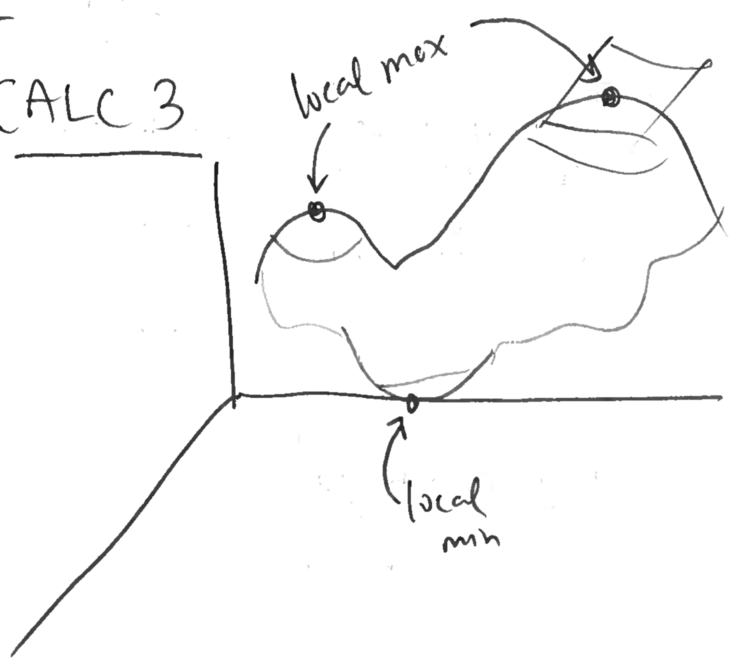
CALC 1



2nd deriv test: if $f'(c)=0$ and $f''(c)>0 \Rightarrow$ local min at c

$f''(c)<0 \Rightarrow$ local max at c

CALC 3



<u>Theorem</u>: If f has a local max or a local min at (a,b), then

$$\boxed{f_x(a,b)=0 \quad \text{and} \quad f_y(a,b)=0}$$

need to look for

Geometric interp: tangent planes are horizontal at local extrema