Illustrative example:

Would like: $\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$

$$\frac{\text{LoDHi} - \text{HiDLo}}{\text{LoLo}} = \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

We know

$$\frac{d}{dx}\left[x^2\right] = 2x$$

$$\frac{d}{dx}\left[x^2\right] = \frac{d}{dx}\left[x \cdot x\right]$$

$$= \frac{d}{dx}[x]\,x + x\,\frac{d}{dx}[x]$$

$$= 1 \cdot x + x \cdot 1$$

$$= 2x$$

Ex: Find eqt of tan line to $H(w) = e^w \sin(w)$ at $w = \frac{\pi}{2}$.

Soln: $H\left(\frac{\pi}{2}\right) = e^{\pi/2}\underbrace{\sin(\pi/2)}_{=1} = e^{\pi/2} \longrightarrow \left(\frac{\pi}{2}, e^{\pi/2}\right)$

$$H'(w) = \frac{d}{dw}\left[e^w \sin(w)\right] \overset{\text{Prod Rule}}{=} \frac{d}{dw}\left[e^w\right]\sin(w) + e^w \frac{d}{dw}\left[\sin(w)\right]$$

$$= e^w \sin(w) + e^w \cos(w)$$

$$H'\left(\frac{\pi}{2}\right) = e^{\pi/2}\underbrace{\sin(\pi/2)}_{=1} + e^{\pi/2}\underbrace{\cos(\pi/2)}_{=0}$$

$$= e^{\pi/2}$$

Tan line

$$\Rightarrow y - e^{\pi/2} = e^{\pi/2}\left(x - \frac{\pi}{2}\right)$$

Let's find derivatives of other 4 trig functs

$$\tan(x) = \frac{\sin(x)}{\cos(x)} \Rightarrow \frac{d}{dx}\tan(x) = \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right]$$

$$= \frac{\cos(x)\frac{d}{dx}[\sin x] - \sin(x)\underbrace{\frac{d}{dx}[\cos(x)]}_{=-\sin(x)}}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

$$= \frac{1}{\cos^2(x)} = \left(\frac{1}{\cos(x)}\right)^2 = \sec^2(x)$$

Pythagorean identity
$\cos^2(x) + \sin^2(x) = 1$

↑ for all x

$$\sec(x) = \frac{1}{\cos(x)} \Rightarrow \frac{d}{dx}\sec(x) = \frac{d}{dx}\left[\frac{1}{\cos(x)}\right]$$

$$= \frac{0 - (-\sin(x))}{\cos^2(x)} = \frac{\sin x}{\cos(x)\cos(x)}$$

$$= \tan(x)\sec(x)$$

$$\csc(x) = \frac{1}{\sin x} \Rightarrow \frac{d}{dx}\csc(x) = \frac{d}{dx}\frac{1}{\sin x}$$

$$= \frac{0 - \cos(x)}{\sin^2(x)} = \frac{-\cos(x)}{\sin(x)\sin(x)} = -\cot(x)\csc(x)$$

$$\cot(x) = \frac{\cos(x)}{\sin(x)} \Rightarrow \frac{d}{dx}\cot(x) = \frac{d}{dx}\left[\frac{\cos(x)}{\sin(x)}\right]$$

$$= \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} = \frac{-[\sin^2(x) + \cos^2(x)]}{\sin^2(x)}$$

$$= -\csc^2(x)$$

<u>Ex</u>: Find $f'(\Omega)$ if $f(\Omega) = \Omega\csc(\Omega) - 2\cos(\Omega)$

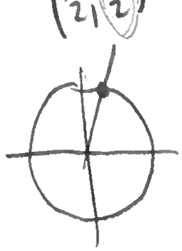
$$f'(\Omega) = \frac{d}{d\Omega}\left[\Omega\csc(\Omega) - 2\cos(\Omega)\right]$$

$$= \frac{d}{d\Omega}\left[\Omega\csc(\Omega)\right] - 2\frac{d}{d\Omega}\left[\cos(\Omega)\right]$$

$$= 1\cdot\csc(\Omega) + \Omega\left(-\cot(\Omega)\csc(\Omega)\right)$$
$$-2(-\sin(\Omega))$$
$$= \csc(\Omega) - \Omega\cot(\Omega)\csc(\Omega) + 2\sin(\Omega)$$

<u>Ex</u>: Let $h(\psi) = \dfrac{2\cos(\psi)}{\psi^2+1}$. Find eqt of tan line at $\psi = \frac{\pi}{3}$.

$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$



<u>Soln</u>: $h(\pi/3) = \dfrac{2\cos(\pi/3)}{(\pi/3)^2+1} = \dfrac{2(\frac{1}{2})}{\pi^2/9+1} = \dfrac{1}{\pi^2/9+1}$

$$\Rightarrow \left(\pi/3, \frac{1}{\pi^2/9+1}\right)$$

$$h'(\psi) = \frac{(\psi^2+1)2[-\sin(\psi)] - 2\cos(\psi)(2\psi)}{(\psi^2+1)^2}$$

$$h'\left(\frac{\pi}{3}\right) = \frac{(\pi^2/9+1)(-2\sin(\pi/3)) - 2\cos(\pi/3)(2\pi/3)}{(\pi^2/9+1)^2} = \frac{\overbrace{-\sqrt{3}(\pi^2/9+1) - \frac{2\pi}{3}}^{\text{a number}}}{(\pi^2/9+1)^2}$$

Tan line: $y - \dfrac{1}{\frac{\pi^2}{9}+1} = \left(\dfrac{-\sqrt{3}(\pi^2/9+1) - \frac{2\pi}{3}}{(\pi^2/9+1)^2}\right)\left[\psi - \dfrac{\pi}{3}\right]$

Ex: Find $\frac{d}{dk}\left[\frac{1}{a+k\sin(a)}\right]$

a is a constant

$$\frac{d}{dk}\left[\frac{1}{a+k\sin(a)}\right] = \frac{0-1(0+\sin(a))}{(a+k\sin(a))^2}$$

$$= \frac{-\sin(a)}{(a+k\sin(a))^2}$$

$\frac{\partial}{\partial k}$ ← "partial derivative w.r.t. k"

Ex: Ohm's Law: $I = \frac{V}{R}$

voltage (volts)

current (amps)

resistance (ohms, Ω)

Suppose voltage + resistance depend on time (hence so does current)

At time $t=3$ (seconds), the voltage is 10v and decreasing at a rate of $0.1\frac{V}{sec}$ and resistance is 30Ω + increasing at a rate of $0.2\frac{\Omega}{s}$.

derivative!

What is rate of change of current at time $t=3$?

derivative

Soln: $I(t) = \frac{V(t)}{R(t)}$

We want $\frac{dI}{dt}$ when $t=3$: $V(3)=10$ $\quad R(3)=30$

$\frac{dV}{dt}(3) = -0.1 \quad \frac{dR}{dt}(3) = 0.2$

$$\frac{dI}{dt} = \frac{d}{dt}\left[\frac{V(t)}{R(t)}\right] = \frac{R(t)V'(t) - V(t)R'(t)}{(R(t))^2}$$

So,

$$\left.\frac{dI}{dt}\right|_{t=3} = \frac{R(3)V'(3) - V(3)R'(3)}{(R(3))^2}$$

$$= \frac{30(-0.1) - 10(0.2)}{(30)^2}$$

$$= \frac{-3-2}{(30)^2} = \frac{-5}{30^2} \ \frac{\text{amps}}{s}$$

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Unit check:

$$I = \frac{V}{R} \qquad \boxed{amp = \frac{volt}{ohm}}$$

$$I' = \frac{RV' - VR'}{R^2} \qquad \frac{amp}{sec} = \frac{ohm\left(\frac{volt}{sec}\right) - volt\left(\frac{ohm}{sec}\right)}{ohm^2}$$

summing units gives the same unit

$$= \frac{\cancel{ohm} \cdot volt}{sec \cdot ohm^{\cancel{2}}} = \frac{volt}{sec \cdot ohm}$$

$$= \frac{amp}{sec}$$