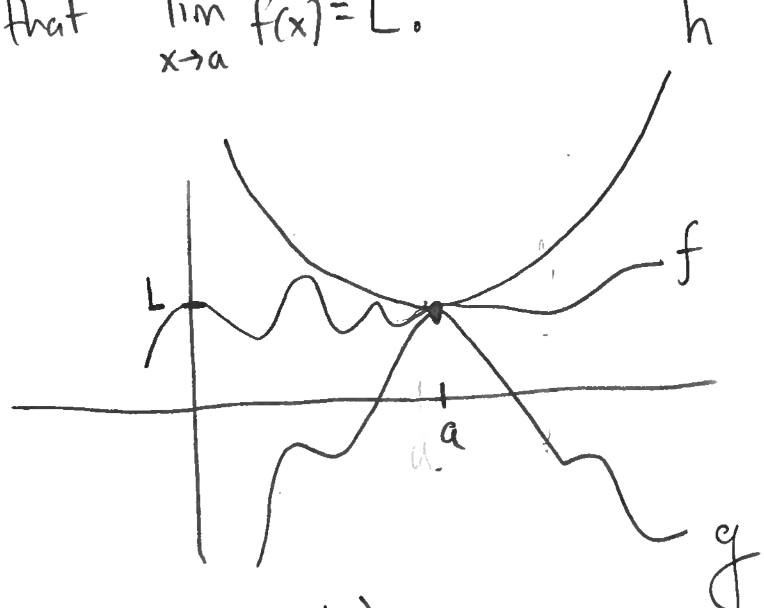
One more thing about limits

Squeeze theorem: if $g(x) \le f(x) \le h(x)$ for x near a

AND if $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then we conclude

that $\lim_{x\to a} f(x) = L$.



Ex: $\lim_{x\to 0} \sin(x)\cos\left(\frac{1}{x^4}\right)$

Soln: From trig: $-1 \le \cos(\theta) \le 1$

$\downarrow \theta = \frac{1}{x^4}$

$a < b$

$\downarrow$ mult by $d<0$

$ad > bd$

$$-1 \le \cos\left(\frac{1}{x^4}\right) \le 1$$

$\sin(x) < 0 \leftarrow x < 0$ (near 0) — mult by $\sin(x)$

$$-\sin(x) \ge \sin(x)\cos\left(\frac{1}{x^4}\right) \ge \sin(x)$$

$\Downarrow$

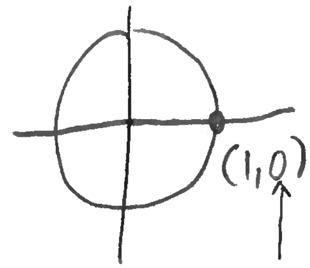
for $x<0$ near 0, $\sin(x) \le \sin(x)\cos\left(\frac{1}{x^4}\right) \le -\sin(x)$

$x > 0$ (near 0) $\rightarrow \sin(x) > 0$

for $x>0$ near 0, $-\sin(x) \le \sin(x)\cos\left(\frac{1}{x^4}\right) \le \sin(x)$

<u>Case I</u>: for $x<0$, near 0:

$$\underbrace{\sin(x)}_{g(x)} \leq \underbrace{\sin(x)\cos\left(\frac{1}{x^4}\right)}_{f(x)} \leq \underbrace{-\sin(x)}_{h(x)}$$



Compute

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \sin(x) = 0$$

$$\lim_{x\to 0^-} h(x) = \lim_{x\to 0^-} -\sin(x) = -\sin(0) = 0$$

Therefore, by Squeeze theorem,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \sin(x)\cos\left(\frac{1}{x^4}\right) = 0$$

<u>Case II</u>: for $x>0$, near 0:

$$\underbrace{-\sin(x)}_{g(x)} \leq \underbrace{\sin(x)\cos\left(\frac{1}{x^4}\right)}_{f(x)} \leq \underbrace{\sin(x)}_{h(x)}$$

$$\lim_{x\to 0^+} g(x) = 0 \qquad \lim_{x\to 0^+} h(x) = 0$$

Therefore, by squeeze thm,

$$\lim_{x\to 0^+} f(x) = 0$$

So, since $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 0$, we may conclude that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \sin(x)\cos\left(\frac{1}{x^4}\right) = 0$$

<u>Ex</u>: $\lim_{x \to 0} x^2 \underbrace{\sin\left(e^{\frac{1}{x}}\right)}$

$e^{\frac{1}{x}} > 0$ for all $x \in \mathbb{R} \setminus \{0\}$

$x = 0$

$0^2 \sin\left(e^{\frac{1}{0}}\right)$

$\sin(\infty)$

more work to do

$$-1 \leq \sin\left(e^{\frac{1}{x}}\right) \leq 1$$

$\downarrow$ mult. by x^2

non-negative

$$\underbrace{-x^2}_{g(x)} \leq \underbrace{x^2 \sin\left(e^{\frac{1}{x}}\right)}_{f(x)} \leq \underbrace{x^2}_{h(x)}$$

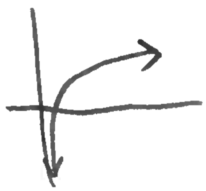
Compute

$\lim_{x \to 0} g(x) = 0$ and $\lim_{x \to 0} h(x) = 0$

Therefore, by Squeeze theorem, can conclude

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x^2 \sin\left(e^{\frac{1}{x}}\right) = 0$$

Ex: $\lim\limits_{x\to 0} x^2\cos(\ln(|x|))$



$x=0$

$0^2\cos(\ln(0))$

$0^2\cos(-\infty)$

$\downarrow$??

more work to do

$$-1 \le \cos(\ln(|x|)) \le 1$$

$$\Big\downarrow \text{ mult by } x^2$$

$$\underbrace{-x^2}_{g(x)} \le \underbrace{x^2\cos(\ln(|x|))}_{f(x)} \le \underbrace{x^2}_{h(x)}$$

$\lim\limits_{x\to 0} g(x) = 0$

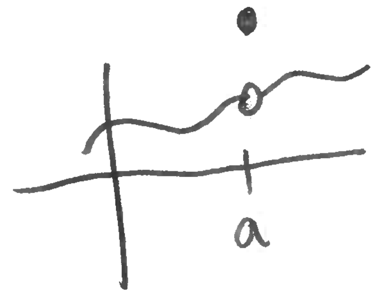
$\lim\limits_{x\to 0} h(x) = 0$

$\therefore$ by squeeze thm,

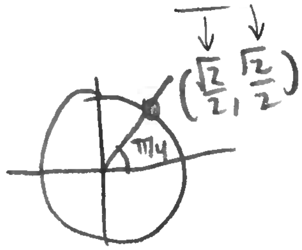
$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2\cos(\ln(|x|)) = 0$$

function f is continuous at $x=a$ whenever:

1. $f(a)$ exists,
2. $\lim_{x\to a} f(x)$ exists, and
3. $\lim_{x\to a} f(x) = f(a)$



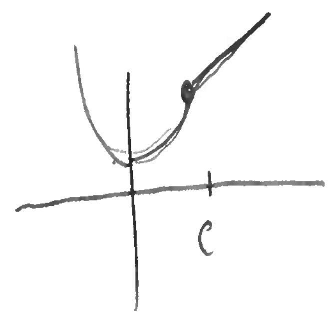
Ex: Use continuity to evaluate



$$\lim_{x\to \pi/4} \tan(x)+3 = \tan(\pi/4)+3$$
$$= 1+3$$
$$=4$$

Ex: Find the values of c that makes

$$f(x) = \begin{cases} x^2+1 & ; \ x \le c \\ x+21 & ; \ x > c \end{cases}$$



Continuous at $x=c$.

Soln: $f(c) = c^2+1$

$\lim_{x\to c^-} f(x) = c^2+1$ $\qquad$ $\lim_{x\to c^+} f(x) = c+21$

Continuity requires: $c^2+1 = c+21 \longrightarrow c^2-c-20=0$

$(c-5)(c+4)=0$

$c=5 \qquad c=-4$

Ex: Find values of x for which

$$f(x) = \frac{5}{x+1} - \frac{2x}{x-10}$$

is not continuous.

PROBLEMS WHEN DENOMS = 0

$x+1=0$ → $\boxed{x=-1}$

$x-10=0$ → $\boxed{x=10}$

f is not continuous at $x=-1$ and $x=10$