"Completing the square"

$$\boxed{x^2 + bx = \left(x+\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2}$$

$$\left(-\frac{3}{2}\right)\left(-\frac{3}{2}\right) = \frac{9}{4}$$

EX: Solve $\boxed{x^2 - 3x} + 8 = 0$

Soln: $\boxed{x^2 - 3x} = x^2 + (-3)x \overset{\text{complete square}}{=} \left(x + \frac{-3}{2}\right)^2 - \left(\frac{-3}{2}\right)^2$

$\boxed{b = -3}$

$$= \left(x - \frac{3}{2}\right)^2 - \frac{9}{4}$$

$$8 = \frac{32}{4}$$

Therefore,

$$\boxed{x^2 - 3x} + 8 = \boxed{\left(x - \frac{3}{2}\right)^2 - \frac{9}{4}} + 8$$

$$= \left(x - \frac{3}{2}\right)^2 + \frac{23}{4} = 0$$

$$\left(x - \frac{3}{2}\right)^2 = -\frac{23}{4}$$

$$x - \frac{3}{2} = \pm\sqrt{-\frac{23}{4}} = \pm\sqrt{\frac{23}{4}}\, i$$

$$= \pm\frac{\sqrt{23}}{2}\, i$$

$$x = \frac{3}{2} \pm \frac{\sqrt{23}}{2}\, i$$

"Solve $x^2 + \_\_ = 0$" ~ "find zeros of $f(x) = x^2 + \_\_$"

~ "Find roots of $f(x) = x^2 + \_\_$"

Ex: Find roots of

$$f(x) = x^2 + x - 10.$$

Soln: We need to solve

$$\boxed{x^2 + x} - 10 = 0.$$

Calculate complete square

$$\boxed{x^2 + x = \left(x + \tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2}$$

$b = 1$

Check

$$\left(x + \tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2$$
$$= x^2 + \tfrac{1}{2}x + \tfrac{1}{2}x + \cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{4}}$$
$$= x^2 + x$$

So,

$$0 = \boxed{x^2 + x} - 10 = \boxed{\left(x + \tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2} - 10$$

$$-\tfrac{1}{4} - 10 = -\tfrac{1}{4} - \tfrac{40}{4}$$

$$0 = \left(x + \tfrac{1}{2}\right)^2 - \tfrac{41}{4}$$

$$\tfrac{41}{4} = \left(x + \tfrac{1}{2}\right)^2$$

$$\pm\sqrt{\tfrac{41}{4}} = x + \tfrac{1}{2}$$

$$x = -\tfrac{1}{2} \pm \sqrt{\tfrac{41}{4}}$$
$$= -\tfrac{1}{2} \pm \tfrac{\sqrt{41}}{2}$$

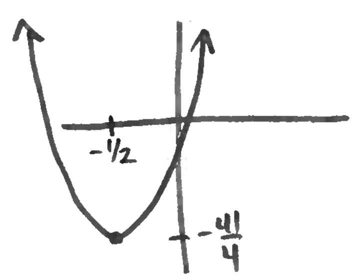
Side note

We have

$$f(x) = x^2 + x - 10$$
$$= \left(x + \tfrac{1}{2}\right)^2 - \tfrac{41}{4}$$

h. shift left

v. shift down



Ex: Solve

$$5x^2 - 8x + 1 = 0$$

$$25x + 5$$
$$= 5(5x+1)$$

Soln: Factor out a 5:

$$0 = 5x^2 - 8x + 1 = 5\left(\boxed{x^2 - \frac{8}{5}x} + \frac{1}{5}\right)$$

$5\star = 8$

$\star = \frac{8}{5}$

$5\star = 1$

$\star = \frac{1}{5}$

compl. square

$$x^2 - \frac{8}{5}x = \left(x - \frac{4}{5}\right)^2 - \left(-\frac{4}{5}\right)^2 = \left(x - \frac{4}{5}\right)^2 - \frac{16}{25}$$

$b = -\frac{8}{5}$

$\frac{b}{2} = \frac{-8}{10} = -\frac{4}{5}$

$$-\frac{16}{25} + \frac{1}{5} = \frac{-16}{25} + \frac{5}{25} = \frac{-11}{25}$$

So,

$$0 = 5\left(x^2 - \frac{8}{5}x + \frac{1}{5}\right) = 5\left(\left[\left(x - \frac{4}{5}\right)^2 - \frac{16}{25}\right] + \frac{1}{5}\right)$$

div by 5

$$0 = \left(x - \frac{4}{5}\right)^2 - \frac{11}{25}$$

$$\frac{11}{25} = \left(x - \frac{4}{5}\right)^2$$

$$\pm\sqrt{\frac{11}{25}} = x - \frac{4}{5}$$

$$\Rightarrow \boxed{x = \frac{4}{5} \pm \frac{\sqrt{11}}{5}}$$

Derive the quadratic formula

We want to solve

$$ax^2+bx+c=0$$

Factor out a:

$$a\left(\boxed{x^2+\frac{b}{a}x}+\frac{c}{a}\right)=0$$

$$x^2+\frac{b}{a}x = \left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2$$

complete square

$$= -\frac{b^2}{4a^2}+\frac{c}{a} = \frac{-b^2+4ac}{4a^2}$$

So,

$$0=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) = a\left(\left(x+\frac{b}{2a}\right)^2\underbrace{-\left(\frac{b}{2a}\right)^2+\frac{c}{a}}\right)$$

$$= a\left(\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right)$$

$\downarrow$ div by a

$$0=\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}$$

$$\frac{b^2-4ac}{4a^2}=\left(x+\frac{b}{2a}\right)^2$$

$$\downarrow$$

$$\pm\sqrt{\frac{b^2-4ac}{4a^2}} = x+\frac{b}{2a} \longrightarrow$$

$$\boxed{\begin{aligned} x &= -\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a} \\ &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}\end{aligned}}$$

cubics

$ax^3+bx^2+cx+d=0$

there is cubic formula
but it's a mess

quartics

$ax^4+bx^3+cx^2+dx+e=0$

HORRIBLE formula
quartic formula

came out of early Renaissance in Italy

1400s

quintic

$ax^5+bx^4+cx^3+dx^2+ex+f=0$

NO quintic formula exists (using $\sqrt{\ }$)

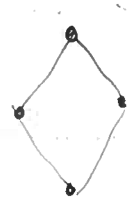
Galois showed no quintic formula exists

(1700s)

died young ~ early 20s

contributions led to group theory

symmetry



methane