①

Recall: $x^2 = 4$ $\quad (-2)^2 = (-2)(-2) = 4$

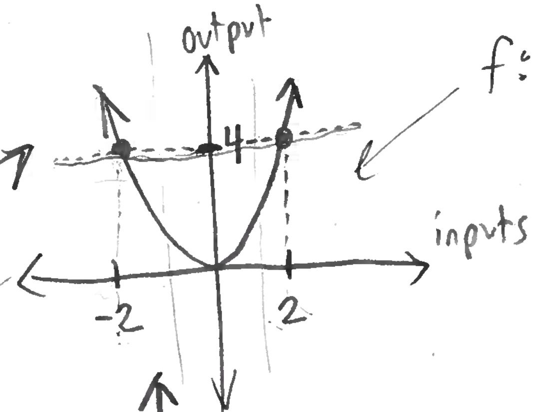
$2^2 = 4$

$\downarrow$

$x = \pm 2$

NOT one-to-one

$f: \mathbb{R} \to \mathbb{R}$



Notice: two different inputs give same output!

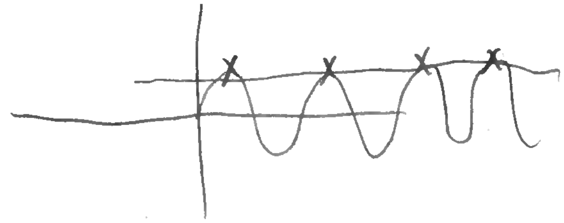
Def: A function $f: X \to Y$ is called **one-to-one** (injective) provided it has property that two different inputs never give same output.

name

domain

codomain (y-axis)

FACT: a one-to-one can be identified if it passes the "horizontal line test" — meaning any horizontal line touches its graph at most once



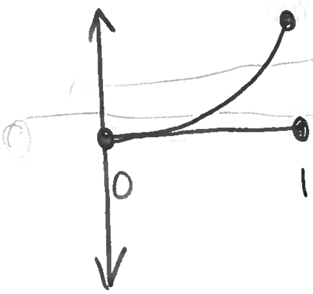
Ex: $f: [0,1] \to \mathbb{R}$

$f(x) = x^2$

$f(x) = 1$

$\downarrow$

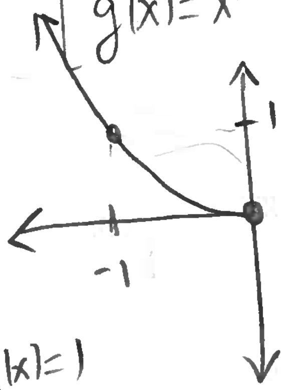
$x = 1$



passes horiz line test!

are one-to-one

Ex: $\begin{cases} g: (-\infty, 0] \to \mathbb{R} \\ g(x) = x^2 \end{cases}$



passes horiz line test!

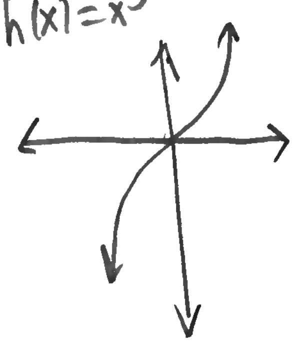
$g(x) = 1$

$\downarrow$

$x = -1$

Ex: $h: \mathbb{R} \to \mathbb{R}$

$h(x) = x^3$



passes h. line test

one-to-one

Inverse functions

Given a function f, we say a function g is an inverse of f if

for all x in dom(g): $f(g(x)) = x$

for all x in dom(f): $g(f(x)) = x$

(g "undoes" f) ~ we usually use notation $f^{-1}(x)$ for "the inverse of f"

Ex: If $f(x) = 2x+1$ then $f^{-1}(x) = \frac{x-1}{2}$.

Test it:

$$f(f^{-1}(x)) = f\left(\frac{x-1}{2}\right)$$

$$= \cancel{2}\left(\frac{x-1}{\cancel{2}}\right) + 1$$

$$= x - 1 + 1 = x \checkmark$$

$$f^{-1}(f(x)) = f^{-1}(2x+1)$$

$$= \frac{(2x+1)-1}{2}$$

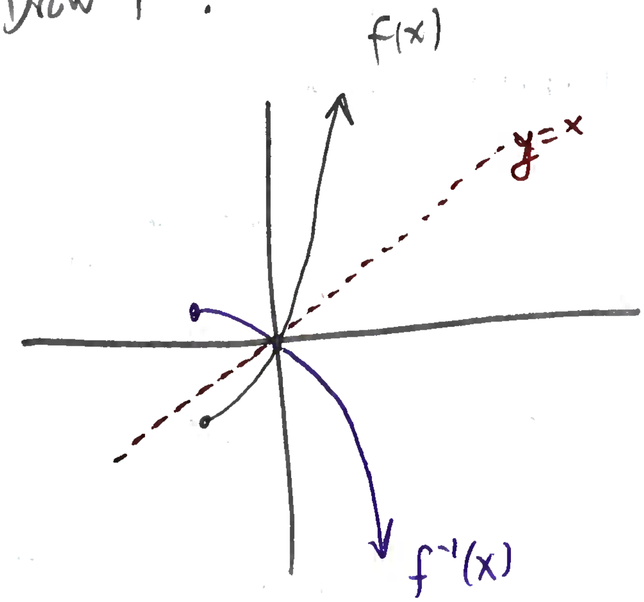
$$= \frac{2x}{2} = x \checkmark$$

$\frac{x-1}{2}$

$x = 2y+1$

FACT: graphs of $y=f(x)$ and $y=f^{-1}(x)$ are mirror images across the line $y=x$ ("symmetric w.r.t. the origin")

Ex: Draw f^{-1}:



Theorem: If f is one-to-one, then f has an inverse.

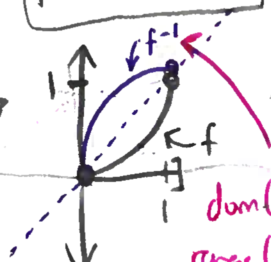
Earlier: we had two one-to-one facts:

$\begin{cases} f: [0,1] \to \mathbb{R} \\ f(x) = x^2 \end{cases}$ $\Downarrow$ $\boxed{f^{-1}(x) = \sqrt{x}}$

$\begin{cases} g: (-\infty, 0] \to \mathbb{R} \\ g(x) = x^2 \end{cases}$ $\Downarrow$ $\boxed{g^{-1}(x) = -\sqrt{x}}$

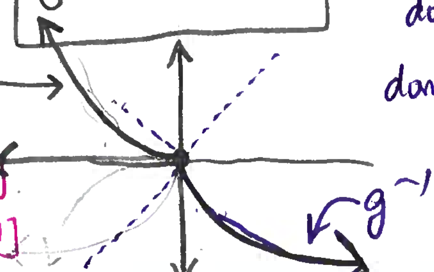
$\sqrt{\ }$ ← "principal square root" always positive

$dom(f) = [0,1]$
$range(f) = [0,1]$



$dom(f^{-1}) = [0,1]$
$range(f^{-1}) = [0,1]$

$range(g) = [0, \infty)$



(in general) inverse does not have same domain as original

$\Downarrow$

NOTICE

$dom(g) = (-\infty, 0]$

$dom(g^{-1}) = [0, \infty)$

What's going on?

<u>Clarity</u>: $f : X \to \text{range}(f)$

then $f^{-1} : \text{range}(f) \to X$