Matrix functions

Recall Calc 2: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

A weird question: what would it mean to take the exp of a matrix ??

We usually define it like this:

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \dots$$

How can it be computed? Depends...

① if $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $A^n = A$ for all n and so

$$\exp\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

$$= \sum \frac{A}{k!}$$

$$= A \sum_{k=0}^{\infty} \frac{1}{k!}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} e$$

$$= \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}$$

is

② if $A = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$ is diagonal. Then

$$A^2 = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} = \begin{bmatrix} d_1^2 & 0 \\ 0 & d_2^2 \end{bmatrix}$$

$$A^3 = \underline{\qquad\qquad} =$$

$$\vdots$$

$$A^n = \begin{bmatrix} d_1^n & 0 \\ 0 & d_2^n \end{bmatrix}, \text{ so}$$

$$\exp\left(\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}\right) = \sum_{k=0}^{\infty} \frac{\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\begin{bmatrix} d_1^k & 0 \\ 0 & d_2^k \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{d_1^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{d_2^k}{k!} \end{bmatrix}$$

$$= \begin{bmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{bmatrix}$$

③ if A is diagonalizable, i.e. $\exists P$ s.t.

$$P^{-1}AP = D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}, \text{ then}$$

$$A = PDP^{-1}, \text{ so}$$

$$\exp(A) = \exp(PDP^{-1}) = \sum \frac{(PDP^{-1})^k}{k!}$$

$$= I + \frac{PDP^{-1}}{1} + \frac{(PDP^{-1})^2}{2!} + \frac{(PDP^{-1})^3}{3!} + \dots$$

BUT,

$$(PDP^{-1})^n = (PDP^{-1})(PDP^{-1})\dots(PDP^{-1}) = PD^nP^{-1},$$

So

$$\exp(A) = \sum_{k=0}^{\infty} \frac{(PDP^{-1})^k}{k!} = \sum_{k=0}^{\infty} \frac{PD^kP^{-1}}{k!}$$

$$= P\left[\sum_{k=0}^{\infty} \frac{D^k}{k!}\right]P^{-1}$$

$$= Pe^{D}P^{-1}$$

Since D is diagonal matrix of e-vals, we can recover A if we know P.

<u>Ex</u>: It can be shown that $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ is diagonalizable and $P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ diagonalizes it:

$$P^{-1} = \left(\frac{1}{2-1}\right)\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

So,

$$P^{-1}AP = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = D$$

Therefore,

$$\exp(A) = P\exp(D)P^{-1}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^2 & 0 \\ 0 & e^3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^2 & e^3 \\ e^2 & e^3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^2 - e^3 & -2e^2 + 2e^3 \\ e^2 - e^3 & -e^2 + 2e^3 \end{bmatrix}$$

④ if A not diagonalizable,,,, other ways!

See "Jordan canonical form"

Other matrix functions   LOG

CALC 2: well-known geo series $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ for $|x| < 1$

$$\int$$

$$-\log(1-x) = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1} = \sum_{k=1}^{\infty} \frac{x^k}{k}$$

$$\Big\downarrow\ x \mapsto -x$$

$$\log(1+x) = -\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k}$$

$$\Big\downarrow\ \text{matrix}$$

$$\log(I+A) = -\sum_{k=1}^{\infty} \frac{(-1)^k A^k}{k!}$$, valid when min {|e-vals of A|} < 1

SINE

or $= \frac{e^{ix} - e^{-ix}}{2i}$

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \xrightarrow{\text{matrix}} \sin(A) = \sum_{k=0}^{\infty} \frac{(-1)^k A^{2k+1}}{(2k+1)!}$$

Ex: $\sin\left(\underbrace{\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}}_{A} \right)$

$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

$P^{-1}AP = D \rightarrow A = PDP^{-1}$

$(PDP^{-1})^{2k+1} = (PDP^{-1}) \cdots (PDP^{-1}) = PD^{2k+1}P^{-1}$

So,

$$\sin\left(\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k P D^{2k+1} P^{-1}}{k!}$$

$$= P \begin{bmatrix} \sin(2) & 0 \\ 0 & \sin(3) \end{bmatrix} P^{-1}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sin(2) & 0 \\ 0 & \sin(3) \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2\sin(2) & \sin(3) \\ \sin(2) & \sin(3) \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2\sin(2) + \sin(3) & -2\sin(2) + 2\sin(3) \\ \sin(2) + \sin(3) & -\sin(2) + 2\sin(3) \end{bmatrix}$$