Homework 2 MATH 3520 Fall 2019

§2.2 #3)

$$\begin{bmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ row echelon and reduced row echelon

#9)

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \overset{r_1 \leftrightarrow r_3}{\sim} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$ ← row echelon

$$\overset{r_1^* = r_1 - r_2}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \overset{r_2^* = r_2 - r_3}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

reduced row echelon

#11)

$$\begin{bmatrix} 3 & 5 \\ 5 & -2 \\ 2 & 4 \end{bmatrix} \underset{r_3^* = r_3 - \frac{2}{3}r_1}{\overset{r_2^* = r_2 - \frac{5}{3}r_1}{\sim}} \begin{bmatrix} 3 & 5 \\ 0 & -31/3 \\ 0 & 2 \end{bmatrix} \overset{r_2^* = -\frac{3}{31}r_2}{\sim} \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ 0 & 2 \end{bmatrix}$$

5 −

$-2 - \frac{5}{3}(5) = -2 - \frac{25}{3}$

$= -\frac{6}{3} - \frac{25}{3}$

$= -\frac{31}{3}$

$4 - \frac{2}{3}(3) = \frac{12}{3} - \frac{6}{3} = \frac{6}{3} = 2$

$$\underset{r_1^* = r_1 - 5r_2}{\overset{r_3^* = r_3 - 2r_2}{\sim}} \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$ ← row echelon

$$\overset{r_1^* = \frac{1}{3}r_1}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$ ← reduced row echelon

②

25)
$$\begin{cases} x_1 + 2x_2 - 3x_3 = 9 \\ 2x_1 - x_2 + x_3 = 0 \\ 4x_1 - x_2 + x_3 = 4 \end{cases}$$

$$\begin{array}{r} 32 \\ \times 5 \\ \hline 160 \end{array} \quad \begin{array}{r} 13 \\ \times 5 \\ \hline 65 \end{array} \quad \begin{array}{r} 18 \\ \times 9 \\ \hline 162 \end{array}$$

$$13 - \frac{9}{5}(7) = \frac{65}{5} - \frac{63}{5} = \frac{2}{5}$$

$$-32 - \frac{9}{5}(-18) = -\frac{160}{5} + \frac{162}{5} = \frac{2}{5}$$

$2 + \frac{2}{5}(-5) = 0$ ✓

$-3 + \frac{2}{5}(7)$

$= -\frac{15}{5} + \frac{14}{5}$

$= -\frac{1}{5}$

$9 + \frac{2}{5}(-18)$

$\frac{45}{5} - \frac{36}{5} = \frac{9}{5}$

$\frac{9}{5} + \frac{1}{5} = \frac{10}{5} = 2$

Soln: Write augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ 4 & -1 & 1 & 4 \end{array}\right] \underset{r_3^* = r_3 - 4r_1}{\overset{r_2^* = r_2 - 2r_1}{\sim}} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 7 & -18 \\ 0 & -9 & 13 & -32 \end{array}\right]$$

$$\overset{r_3^* = r_3 - \frac{9}{5}r_2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 7 & -18 \\ 0 & 0 & 2/5 & 2/5 \end{array}\right]$$

$$\underset{r_3^* = \frac{5}{2}r_3}{\overset{r_1^* = r_1 + \frac{2}{5}r_2}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & -1/5 & 9/5 \\ 0 & -5 & 7 & -18 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\underset{r_1^* = r_1 + \frac{1}{5}r_3}{\overset{r_2^* = r_2 - 7r_3}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -5 & 0 & -25 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\overset{r_2^* = -\frac{1}{5}r_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Therefore the solution of the system is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

#32)
$$\begin{cases} \sqrt{2}x + y + 2z = 1 \\ \sqrt{2}y - 3z = -\sqrt{2} \\ -y + \sqrt{2}z = 1 \end{cases}$$

$$\sqrt{2} - \frac{3}{\sqrt{2}} = \frac{2}{\sqrt{2}} - \frac{3}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Soln: Write aug matrix:

$$\left[\begin{array}{ccc|c} \sqrt{2} & 1 & 2 & 1 \\ 0 & \sqrt{2} & -3 & -\sqrt{2} \\ 0 & -1 & \sqrt{2} & 1 \end{array}\right] \underset{r_3^* = r_3 + \frac{1}{\sqrt{2}} r_2}{\sim} \left[\begin{array}{ccc|c} \sqrt{2} & 1 & 2 & 1 \\ 0 & \sqrt{2} & -3 & -\sqrt{2} \\ 0 & 0 & 1/\sqrt{2} & 0 \end{array}\right]$$

$$\underset{r_3^* = \sqrt{2} r_3}{\sim} \left[\begin{array}{ccc|c} \sqrt{2} & 1 & 2 & 1 \\ 0 & \sqrt{2} & -3 & -\sqrt{2} \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\underset{\substack{r_1^* = r_1 - 2r_3 \\ r_2^* = r_2 + 3r_3}}{\sim} \left[\begin{array}{ccc|c} \sqrt{2} & 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 & -\sqrt{2} \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\underset{r_2^* = \frac{1}{\sqrt{2}} r_2}{\sim} \left[\begin{array}{ccc|c} \sqrt{2} & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\underset{r_1^* = r_1 - r_2}{\sim} \left[\begin{array}{ccc|c} \sqrt{2} & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\underset{r_1^* = \frac{1}{\sqrt{2}} r_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2/\sqrt{2} \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

note:
$$\frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}}\underbrace{\left(\frac{\sqrt{2}}{\sqrt{2}}\right)}_{=1} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

Wolfram alpha simplifies like that!

So the solution is

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} \\ -1 \\ 0 \end{bmatrix}$$

§2.3

#1) $\vec{v}=\begin{bmatrix}1\\2\end{bmatrix}$, $\vec{u}_1=\begin{bmatrix}1\\-1\end{bmatrix}$, $\vec{u}_2=\begin{bmatrix}2\\-1\end{bmatrix}$

Soln: "Is $\vec{v}$ a linear combo of $\vec{u}_1$ and $\vec{u}_2$?"

means

"Can we find scalars α_1, α_2 so that

$$\boxed{\alpha_1\vec{u}_1+\alpha_2\vec{u}_2 = \vec{v}}$$?"

"a linear combo of $\vec{u}_1$ and $\vec{u}_2$"

$$\alpha_1\underbrace{\begin{bmatrix}1\\-1\end{bmatrix}}_{\vec{u}_1}+\alpha_2\underbrace{\begin{bmatrix}2\\-1\end{bmatrix}}_{\vec{u}_2}=\underbrace{\begin{bmatrix}1\\2\end{bmatrix}}_{\vec{v}}$$

$\downarrow$ vector algebra

$$\begin{bmatrix}\alpha_1+2\alpha_2\\-\alpha_1-\alpha_2\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$$

$\downarrow$ same as the system of eqts

$$\begin{cases}\alpha_1+2\alpha_2=1\\-\alpha_1-\alpha_2=2\end{cases}$$

$\downarrow$ has aug matrix

I recommend jumping from linear combo straight to aug matrix!!!

$$\left[\begin{array}{cc|c}1&2&1\\-1&-1&2\end{array}\right]\overset{r_2^*=r_2+r_1}{\sim}\left[\begin{array}{cc|c}1&2&1\\0&1&3\end{array}\right]\overset{r_1^*=r_1-2r_2}{\sim}\left[\begin{array}{cc|c}1&0&-5\\0&1&3\end{array}\right]$$

$\Rightarrow$ System has soln

$$\vec{\alpha}=\begin{bmatrix}\alpha_1\\\alpha_2\end{bmatrix}=\begin{bmatrix}-5\\3\end{bmatrix}$$

$\Rightarrow$ YES — $\vec{v}$ is a linear combo of $\vec{u}_1$ and $\vec{u}_2$

#4] $\vec{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

5

Soln: Goal is to find scalars α_1, α_2 s.t.

$$\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 = \vec{v}$$, i.e.

$$\alpha_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

$\downarrow$ Aug matrix

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & -1 \end{array}\right] \overset{r_2^* = r_2 - r_1}{\sim} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{array}\right]$$

$$\overset{r_3^* = r_3 - r_2}{\sim} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ soln vector is

$$\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$\Rightarrow$ yes $\vec{v} = 3\vec{u}_1 - \vec{u}_2$

#8) is $\vec{b} = \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}$ in the span of the columns of $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{bmatrix}$?

<u>Soln</u>: Anything in the span of the columns of A is a linear combo of cols of A, i.e., there exist scalars $\alpha_1, \alpha_2, \alpha_3$ s.t.

$$\alpha_1 \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix} + \alpha_3 \begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix}$$

$\Big\downarrow$ Aug mat

2 12
32
24
8

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}\right] \underset{r_3^* = r_3 - 9r_1}{\overset{r_2^* = r_2 - 5r_1}{\sim}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -24 \end{array}\right]$$

$$\overset{r_3^* = r_3 - 2r_2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

<u>NOT</u> a problem! encodes "$0=0$"

$$\overset{r_1^* = r_1 + \frac{1}{2}r_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\overset{r_2^* = -\frac{1}{4}r_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{matrix} \alpha_1 - \alpha_3 = -2 \\ \alpha_2 + 2\alpha_3 = 3 \end{matrix}$$

$$\Rightarrow \begin{matrix} \alpha_1 = -2 + \alpha_3 \\ \alpha_2 = 3 - 2\alpha_3 \end{matrix}$$

From here, we can conclude <u>YES</u>, $\vec{b}$ is in span of cols of A.

More precisely, we know there's ∞ ways to find the $\alpha_1, \alpha_2, \alpha_3$: soln of system is

$$\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} -2 + \alpha_3 \\ 3 - 2\alpha_3 \\ \alpha_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \alpha_3}$$

picking any scaler α_3 will yield a linear combo of cols of A that equals $\vec{b}$

Problem A: Solve
$$\begin{cases} iz_1 + 2iz_2 = 1 \\ -z_1 + iz_2 = 0 \end{cases}$$

Soln; Write aug matrix:

$$\left[\begin{array}{cc|c} i & 2i & 1 \\ -1 & i & 0 \end{array}\right] \underset{r_2^* = r_2 + ir_1}{\sim} \left[\begin{array}{cc|c} i & 2i & 1 \\ 0 & i+2 & -i \end{array}\right]$$

$$\underset{r_2^* = \frac{1}{i+2} r_2}{\sim} \left[\begin{array}{cc|c} i & 2i & 1 \\ 0 & 1 & \frac{-i}{i+2} \end{array}\right]$$

$$\underset{r_1^* = r_1 - 2ir_2}{\sim} \left[\begin{array}{cc|c} i & 0 & 1 + \frac{2}{i+2} \\ 0 & 1 & \frac{-i}{i+2} \end{array}\right]$$

$$\underset{r_1^* = \frac{1}{i} r_1}{\sim} \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{i} + \frac{2}{-1+2i} \\ 0 & 1 & \frac{-i}{i+2} \end{array}\right]$$

$1 - 2i\left(\frac{i}{i+2}\right)$

$= 1 + \frac{2}{i+2}$

$1 - 2i\left(\frac{-i}{i+2}\right)$

$1 - \frac{2}{i+2}$

Therefore, the solution is

$$\vec{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{i} + \frac{2}{-1+2i} \\ -\frac{i}{i+2} \end{bmatrix}$$