MATH 3520 - EXAM 3 FALL 2019

SOLUTION

Friday, 22 November
Instructor: Tom Cuchta

Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (10 points) If $T\colon \mathbb{R}^{2\times 1} \to \mathcal{P}_2$ is a linear transformation and

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = x+3 \quad \text{and} \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = x^2-5,$$

then find $T\left(\begin{bmatrix}2\\-3\end{bmatrix}\right)$.

Solution: We must express the vector $\begin{bmatrix}2\\-3\end{bmatrix}$ in terms of the vectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$: consider the equation

$$\alpha_1\begin{bmatrix}1\\1\end{bmatrix} + \alpha_2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2\\-3\end{bmatrix}.$$

We can solve this system of equations using an augmented matrix:

$$\begin{bmatrix}1 & 0 & 2\\ 1 & 1 & -3\end{bmatrix} \overset{\text{rref}}{\sim} \begin{bmatrix}1 & 0 & 2\\ 0 & 1 & -5\end{bmatrix}.$$

Therefore,

$$\begin{aligned} T\left(\begin{bmatrix}2\\-3\end{bmatrix}\right) &= T\left(2\begin{bmatrix}1\\1\end{bmatrix} - 5\begin{bmatrix}0\\1\end{bmatrix}\right) \\ &\overset{T \text{ is linear transf.}}{=} 2T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) - 5T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \\ &= 2(x+3) - 5(x^2-5) \\ &= 2x + 6 - 5x^2 + 25 \\ &= -5x^2 + 2x + 31. \end{aligned}$$

2. (10 points) Define the linear transformation $T\colon \mathcal{P}_2 \to \mathcal{P}_2$ given by $T(p) = x^2p''(x)$. Find range(T).
Solution: Let $p = ax^2 + bx + c$. Compute

$$\begin{aligned} T(p) &= x^2p''(x) \\ &= x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(ax^2+bx+c\right) \\ &= x^2(2a) \\ &= 2ax^2. \end{aligned}$$

Therefore,

$$\text{range}(T) = \{2ax^2 \colon a \in \mathbb{R}\}.$$

3. (18 points) (**Laguerre polynomials**) Consider the linear transformation $T\colon \mathcal{P}_2 \to \mathcal{P}_2$ given by $T(p) = xp'' + (1-x)p' + p$.
   (a) (7 points) Find $\dim(\mathcal{P}_2)$.
   Solution: Since $\mathcal{P}$ has a basis $\mathscr{B} = \{1, x, x^2\}$, we see that

   $$\dim\mathcal{P}_2 = 3.$$

   (b) (7 points) Find $\ker(T)$.
   Solution: To find the kernel, we need to determine which polynomials $p = ax^2 + bx + c \in \mathcal{P}_2$ map to the zero polynomial, so consider

   $$\begin{aligned} 0x^2 + 0x + 0 &\overset{\text{set}}{=} T(p) \\ &= x\frac{\mathrm{d}^2}{\mathrm{d}x^2}(ax^2+bx+c) + (1-x)\frac{\mathrm{d}}{\mathrm{d}x}(ax^2+bx+c) + (ax^2+bx+c) \\ &= x(2a) + (1-x)(2ax+b) + (ax^2+bx+c) \\ &= 2ax + 2ax + b - 2ax^2 - bx + ax^2 + bx + c \\ &= \Big(-2a+a\Big)x^2 + \Big(2a+2a-b+b\Big)x + \Big(b+c\Big) \\ &= -ax^2 + 4ax + (b+c). \end{aligned}$$

Now equating coefficients yields the system

$$\begin{aligned} -a &= 0 \\ 4a &= 0 \\ b + c &= 0, \end{aligned}$$

hence $a = 0$ and $b = -c$. Therefore, the kernel of T is

$$\ker(T) = \{-cx + c \colon c \in \mathbb{R}\},$$

which has basis $\mathscr{B} = \{-x + 1\}$.

(c) (4 points) Using the previous parts and the rank-nullity theorem, find rank(T).
Solution: Since $\dim(\mathcal{P}_2) = 3$ and nullity$(T) = \dim \ker T = 1$, the rank-nullity theorem tells us

$$3 = 1 + \text{rank}(T),$$

hence

$$\text{rank}(T) = 2.$$

4. (8 points) Given that for some matrices $A, B \in \mathbb{R}^{n\times n}$, $\det(A) = 3$ and $\det(B) = -1$, compute

(a) (4 points) $\det(AB)$
Solution:

$$\det(AB) = \det(A)\det(B) = 3(-1) = -3$$

(b) (4 points) $\det\left(A^{-1}B^{-1}\right)$
Solution:

$$\det(A^{-1}B^{-1}) = \det(A^{-1})\det(B^{-1}) = \frac{1}{\det(A)\det(B)} = -\frac{1}{3}.$$

5. (18 points) Consider the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 4 \end{bmatrix}$.

(a) (6 points) Find the eigenvalues of A. (*hint: you should find three*)
Solution: Consider the characteristic equation

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \det\left(\begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & -1 & 4-\lambda \end{bmatrix}\right) \\ &= (1-\lambda)\det\left(\begin{bmatrix} 2-\lambda & 0 \\ -1 & 4-\lambda \end{bmatrix}\right) - 0 + 0 \\ &= (1-\lambda)\Big((2-\lambda)(4-\lambda) - 0\Big) \\ &= (1-\lambda)(2-\lambda)(4-\lambda), \end{aligned}$$

therefore we get three eigenvalues: $\lambda = 1, 2, 4$.

(b) (6 points) Find an eigenvector associated to ***only the smallest*** eigenvalue you found in the previous part.

Solution: We consider the equation $A\vec{x} = 1\vec{x}$, i.e. $A\vec{x} = \vec{x}$, where $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Written out, this becomes

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Expanding the left-hand side yields the vector equation

$$\begin{bmatrix} x_1 + x_2 \\ 2x_2 \\ -x_2 + 4x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Subtracting the vector on the right yields

$$\begin{bmatrix} x_2 \\ x_2 \\ -x_2 + 3x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, we may solve the system using row reduction:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & 0 \end{bmatrix} \overset{\text{rref}}{\sim} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

meaning that $x_2 = x_3 = 0$. Therefore taking $x_1 = 1$ yields an eigenvector

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

(c) (6 points) Find the eigenspace of ***only the smallest*** eigenvalue you found earlier.
Solution: From the previous part, we see that the eigenspace is of the form

$$E_1 = \left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} : x_1 \in \mathbb{R} \right\}.$$

6. (12 points) Consider the matrix $P = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$.

(a) (6 points) Find P^{-1}. $\left(\textit{hint: recall the inverse of } M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \textit{ is the matrix } \frac{1}{\det(M)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)$

Solution:

$$P^{-1} = \frac{1}{\det P} \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

(b) (6 points) Diagonalize the matrix $A = \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix}$ using the matrix P.

Solution: Calculate

$$\begin{aligned} P^{-1}AP &= \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 6 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -4 & 4 \\ 10 & -5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix} \end{aligned}$$

7. (8 points) Consider the inner product space $(\mathcal{P}_2, \langle \cdot, \cdot \rangle)$ where $\langle p, q \rangle = \int_0^1 p(x)q(x)\mathrm{d}x$.

(a) (6 points) Calculate $\langle 1, 4x^3 - 2x \rangle$
Solution: Calculate

$$\begin{aligned} \langle 1, 4x^3 - 2x \rangle &= \int_0^1 4x^3 - 2x\mathrm{d}x \\ &= x^4 - x^2 \Big|_0^1 \\ &= 1 - 1 \\ &= 0. \end{aligned}$$

(b) (2 points) Based on your answer above, is 1 orthogonal to $4x^3 - 2x$?
Solution: Yes – the inner product was equal to zero.

8. (18 points) Consider the inner product space $(\mathcal{P}, \langle .,. \rangle)$, where $\langle p, q\rangle = \int_0^\infty p(x)q(x)\sin(x)e^{-x}\mathrm{d}x$. In this inner product space, the moments are $\langle 1, 1\rangle = \frac{1}{2}$, $\langle 1, x\rangle = \frac{1}{2}$, $\langle 1, x^2\rangle = \frac{1}{2}$, $\langle 1, x^3\rangle = 0$, and $\langle 1, x^4\rangle = -3$.

(a) (9 points) Compute $\langle 2x^4 - 3x^3 + x^2 - 1, 1\rangle$.
Solution: Using the "linear in the first argument" property of inner products multiple times, compute

$$\begin{aligned}
\langle 2x^4 - 3x^3 + x^2 - 1, 1\rangle &= 2\langle x^4, 1\rangle - 3\langle x^3, 1\rangle + \langle x^2, 1\rangle - \langle 1, 1\rangle \\
&= 2(-3) - 3(0) + \frac{1}{2} - \frac{1}{2} \\
&= -6
\end{aligned}$$

(b) (9 points) Compute $\mathrm{proj}_x(x^2 - 1)$.
Solution: Compute

$$\begin{aligned}
\mathrm{proj}_x(x^2 - 1) &= \frac{\langle x, x^2 - 1\rangle}{\langle x, x\rangle} x \\
&\overset{\text{inner prod props}}{=} \frac{\langle 1, x^3\rangle - \langle 1, x\rangle}{\langle 1, x^2\rangle} x \\
&= \frac{0 - \frac{1}{2}}{\frac{1}{2}} x \\
&= -x
\end{aligned}$$