MATH 3520 - EXAM 2 FALL 2019

SOLUTION

Friday, 18 October
Instructor: Tom Cuchta

Instructions:

- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

1. (12 points) Is W a subspace of V? If so, prove it. If not, explain why not (preferably using a counterexample).

   (a) (6 points) $V = \mathscr{P}_2$ and $W = \{at^2 + b\colon a, b \in \mathbb{R}\}$
   Solution: Yes W is a subspace of V. First we show that it is closed under vector addition: let $at^2 + b, ct^2 + d \in W$. Then,
   $$(at^2 + b) + (ct^2 + d) = (a + c)t^2 + (b + d) \in W,$$
   because $a + c \in \mathbb{R}$ and $b + d \in \mathbb{R}$. Now we show that it is closed under scalar multiplication: let $at^2 + b \in W$ and let $\alpha \in \mathbb{R}$ be a scalar, then
   $$\alpha(at^2 + b) = \alpha a t^2 + \alpha b \in W,$$
   because $\alpha a \in \mathbb{R}$ and $\alpha b \in \mathbb{R}$. Therefore W is a subspace of V.

   (b) (6 points) $V = \mathbb{R}^{2\times 2}$ and $W = \left\{ \begin{bmatrix} a & 0 \\ 1 & b \end{bmatrix} : a, b \in \mathbb{R} \right\}$
   Solution: No, W is not a subspace of V because it is not closed under addition. For instance, let $\vec{v}_1 = \vec{v}_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \in W$. Then
   $$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \notin W,$$
   because it is has a "2" in the lower left corner, contrary to the condition that defines W.

2. (11 points) Determine whether or not $\mathscr{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for vector space $V = \mathbb{R}^{3\times 1}$. Explain why or why not.

   Solution: It is a basis. To check it, first check whether or not $\mathscr{B}$ spans $\mathbb{R}^{3\times 1}$: let $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^{3\times 1}$, and consider the vector equation
   $$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$
   This becomes
   $$\begin{bmatrix} \alpha_1 + \alpha_3 \\ \alpha_2 \\ -\alpha_1 + \alpha_2 + \alpha_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$
   We will know that $\mathscr{B}$ spans $\mathbb{R}^{3\times 1}$ if we can solve this system of linear equations. Of course we can write it as an augmented matrix and put it into reduced echelon form:
   $$\begin{bmatrix} 1 & 0 & 1 & a \\ 0 & 1 & 0 & b \\ -1 & 1 & 1 & c \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \frac{a+b-c}{2} \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & \frac{a-b+c}{2} \end{bmatrix},$$
   meaning that we have solution vector
   $$\vec{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \frac{a+b-c}{2} \\ b \\ \frac{a-b+c}{2} \end{bmatrix},$$
   and so we know that $\mathscr{B}$ spans $\mathbb{R}^{3\times 1}$. Now we must check if $\mathscr{B}$ is linearly independent. This means we must check to see if there are any nonzero solutions to the vector equation
   $$\alpha_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

However, we have already solved this equation and so we simply plug in $a = b = c = 0$ into our earlier solution and we observe solution vector

$$\vec{\alpha} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore $\mathscr{B}$ is linearly independent. Since $\mathscr{B}$ is linearly independent and spans $\mathbb{R}^{3\times 1}$, we conclude that it is a basis for $\mathbb{R}^{3\times 1}$.

3. (16 points) Consider the basis $\mathscr{B} = \{1, 2x, 4x^2 - 2\}$ of $\mathscr{P}_2$ (this basis for $\mathscr{P}_2$ is called the "Hermite polynomials"). Find $[x^2 + x + 1]_{\mathscr{B}}$.
Solution: We have to express the polynomial x^2+x+1 in terms of the basis $\mathscr{B}$. This amounts to finding weights α_1, α_2, and α_3 so that

$$x^2 + x + 1 = \alpha_1(1) + \alpha_2(2x) + \alpha_3(4x^2 - 2).$$

Rearranging the right-hand side yields

$$x^2 + x + 1 = \Big(4\alpha_3\Big)x^2 + \Big(2\alpha_2\Big)x + \Big(\alpha_1 - 2\alpha_3\Big).$$

Equating coefficients yields the system of linear equations

$$\begin{cases} 4\alpha_3 & = 1 \\ 2\alpha_2 & = 1 \\ \alpha_1 - 2\alpha_3 & = 1. \end{cases}$$

We can solve this directly or with an augmented matrix:

$$\begin{bmatrix} 0 & 0 & 4 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & -2 & 1 \end{bmatrix} \overset{\text{rref}}{\sim} \begin{bmatrix} 1 & 0 & 0 & \dfrac{3}{2} \\ 0 & 1 & 0 & \dfrac{1}{2} \\ 0 & 0 & 1 & \dfrac{1}{4} \end{bmatrix}.$$

Therefore, we have shown that

$$[x^2 + x + 1]_{\mathscr{B}} = \begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix}.$$

4. (17 points) Consider the basis $\mathscr{B} = \left\{1, x - \dfrac{1}{2}, x^2 - x\right\}$ (called "Euler polynomials") and the basis $\mathscr{C} = \left\{1, x, x^2\right\}$ of $\mathscr{P}_2$. Find the change of basis matrix $P_{\mathscr{C}\leftarrow\mathscr{B}}$.
Solution: The change of basis matrix is given by

$$P_{\mathscr{C}\leftarrow\mathscr{B}} = \begin{bmatrix} [1]_{\mathscr{C}} & \left[x - \frac{1}{2}\right]_{\mathscr{C}} & \left[x^2 - x\right]_{\mathscr{C}} \end{bmatrix},$$

so we must find those coordinates. To find $[1]_{\mathscr{C}}$, we have to find scalars $\alpha_1, \alpha_2, \alpha_3$ so that

$$1 = \alpha_1(1) + \alpha_2(x) + \alpha_3(x^2).$$

It is immediately clear that $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = 0$. Therefore,

$$[1]_{\mathscr{C}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

To find $\left[x - \frac{1}{2}\right]_{\mathscr{C}}$ we must find scalars $\beta_1, \beta_2, \beta_3$ so that

$$x - \frac{1}{2} = \beta_1(1) + \beta_2(x) + \beta_3(x^2).$$

It is immediately clear that $\beta_1 = -\dfrac{1}{2}$, $\beta_2 = 1$, and $\beta_3 = 0$. Therefore the coordinates are

$$\left[x - \frac{1}{2}\right]_{\mathscr{C}} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}.$$

Finally, to find the coordinates $[x^2 - x]_{\mathscr{C}}$, we must find scalars $\gamma_1, \gamma_2, \gamma_3$ so that

$$x^2 - x = \gamma_1(1) + \gamma_2(x) + \gamma_3(x^2).$$

It is immediately clear that $\gamma_1 = 0$, $\gamma_2 = -1$, and $\gamma_3 = 1$, and so we write

$$[x^2 - x]_{\mathscr{C}} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

Therefore we have found the change of basis matrix:

$$P_{\mathscr{C} \leftarrow \mathscr{B}} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

5. (22 points) Find the dimension of the described subspace W of V.

   (a) (11 points) $V = \mathscr{P}_2, \quad W = \{p \in \mathscr{P}_2 \colon \frac{x^2}{2} p''(x) = p(x)\}$

   Solution: We need to find a basis for W. To do so, consider an arbitrary $p \in \mathscr{P}_2$ given by $p = ax^2 + bx + c$. Then $p'' = 2a$ and so the equation $\frac{x^2}{2} p''(x) = p(x)$ yields

   $$ax^2 = ax^2 + bx + c,$$

   or equivalently

   $$0 = bx + c.$$

   Equating coefficients on each side (note: left-side is same as $0x + 0$) yields $b = 0$ and $c = 0$. Therefore,

   $$W = \{ax^2 + bx + c \colon a \in \mathbb{R}, b = c = 0\} = \left\{ax^2 \colon a \in \mathbb{R}\right\}.$$

   A basis for W could be $\mathscr{B} = \{x^2\}$ and so $\dim(W) = 1$.

   (b) (11 points) $V = \mathscr{P}_2, W = \{A \in \mathbb{R}^{2\times 2} \colon A^T = A\}$

   Solution: Let $A \in \mathbb{R}^{2\times 2}$ be given by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The condition $A^T = A$ implies that

   $$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

   Therefore we see that $a = a$ (useless equation), $c = b$ (useful), $b = c$ (useful), and $d = d$ (useless). This means that the top right and bottom left entry must agree in order to satisfy $A^T = A$. Therefore

   $$W = \left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \colon a, b, d \in \mathbb{R} \right\}.$$

   A basis for W could be $\mathscr{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$, and hence $\dim(W) = 3$.

6. (22 points) Is the function a linear transformation? If so, prove it. If not, explain why not (preferably with a counterexample).

   (a) (11 points) $\begin{cases} T\colon \mathbb{R}^{2\times 1} \to \mathscr{P}_2 \\ T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = abx^2 + b \end{cases}$

   Solution: Not a linear transformation.

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = 4x^2 + 2,$$

   while

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = (x^2+1) + (x^2+1) = 2x^2 + 2 \neq T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right).$$

   This demonstrates that the mapping T is not a linear transformation.

   (b) (11 points) Let V be an arbitrary vector space with zero vector $\vec{0}$: $\begin{cases} T\colon V \to \{\vec{0}\} \\ T(v) = \vec{0} \end{cases}$

   Solution: Let $\vec{v} \in V$. Then,

$$T(\vec{v} + \vec{v}) = T(2\vec{v}) = \vec{0} = T(\vec{v}) + T(\vec{v})),$$

   and

$$T(\alpha\vec{v}) = \vec{0} = \alpha T(\vec{v}).$$

   Therefore this map is a linear transformation.

7. (**BONUS: 5 points**) Recall that a sequence is a function whose domain is the integers. Typically, if a is a sequence, then $a(n)$ is denoted by a_n. For exmaple, the sequence $a_n = n^2$ has values $a_{-2} = 4$, $a_0 = 0$, $a_1 = 1$, etc.

   Let V denote the set of real-valued sequences with the property that for all integers n, $1 + a_n > 0$. Define the addition operation of V, denoted by $\oplus$, to be defined by

$$a_n \oplus b_n = a_n + b_n + a_n b_n.$$

   Define scalar multiplication on V, denoted by $\odot$, by

$$\alpha \odot a_n = \alpha a_n \int_0^1 (1 + a_n x)^{\alpha - 1} \mathrm{d}x.$$

   This structure forms a vector space called the "regressive vector space of the time scale of integers". Consider the sequences $a_n = n^3$ and $b_n = n^2$ and compute $a_n \oplus b_n$ and compute $2 \odot a_n$.
   Solution: By definition,

$$a_n \oplus b_n = n^3 + n^2 + n^3 n^2 = n^3 + n^2 + n^5,$$

   and

$$2 \odot a_n = 2n^3 \int_0^1 (1 + n^2 x)^1 \mathrm{d}x = 2n^3 \left[x + \frac{n^2 x^2}{2} \right|_{x=0}^{x=1} = 2n^3 \left(1 + \frac{n^2}{2}\right).$$