MATH 3504 - EXAM 1 FALL 2019 SOLUTION

Friday, 15 February 2019
Instructor: Tom Cuchta

Instructions:

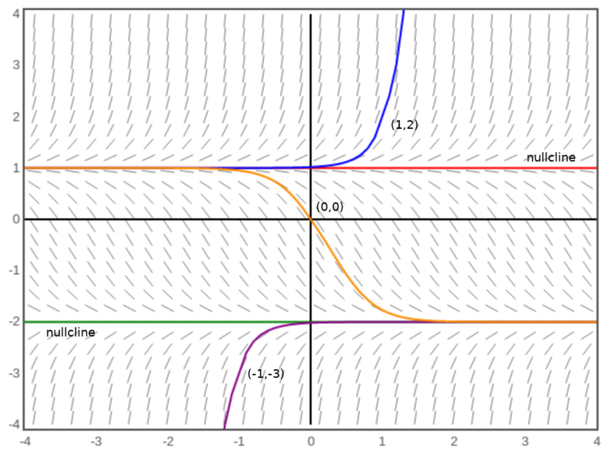
- Show all work, clearly and in order, if you want to get full credit. If you claim something is true **you must show work backing up your claim**. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit.
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point.
- Good luck!

Undetermined coefficients table to solve $ax'' + bx' + cx = f(t)$

Form of $f(t)$	Trial form for $x_p(t)$
α	A
$\alpha e^{\beta t}$	$Ae^{\beta t}$
Polynomial of degree n	$A_n t^n + A_{n-1}t^{n-1} + \ldots + A_1 t + A_0$
$\alpha \sin(\omega t)$	$A\sin(\omega t) + B\cos(\omega t)$
$\alpha \cos(\omega t)$	$A\sin(\omega t) + B\cos(\omega t)$
$\alpha e^{rt}\sin(\omega t)$	$e^{rt}(A\sin(\omega t) + B\cos(\omega t))$
$\alpha e^{rt}\cos(\omega t)$	$e^{rt}(A\sin(\omega t) + B\cos(\omega t))$

1. (10 points) The folowing image is the slope field for the first order nonlinear differential equation

$$x' = (x+2)(x-1).$$



   (a) (4 points) Find the nullcline(s) (i.e. set $x' = 0$) of the differential equation, write them here, and plot them on the image (label your plot(s) as "nullcline").
   Solution: The nullcline is given by $0 = (x+2)(x-1)$. Solve this for x to see $x = -2$ and $x = 1$, which are horizontal lines at heights -2 and 1, respectively.

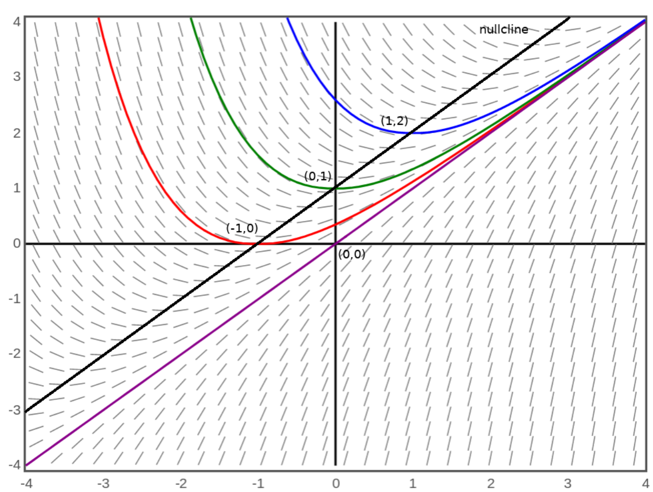
   (b) (2 points) Plot the point $(0,0)$ and sketch the solution through that point.

   (c) (2 points) Plot the point $(1,2)$ and sketch the solution through that point.

   (d) (2 points) Plot the point $(-1,-3)$ and sketch the solution through that point.

2. (10 points) The following image is the slope field for the first order nonlinear differential equation

$$x' = -x + t + 1.$$



   (a) (2 points) Find the nullcline(s) of the differential equation, write them here, and plot them on the image (label your plot(s) as "nullcline").
   Solution: The nullcline is given by $0 = -x + t + 1$, in other words, $x = t + 1$ – the function $x = t$ shifted up by 1.

   (b) (2 points) Sketch the solution curve that travels through the origin $(0,0)$ on the image.

(c) (2 points) Plot the point $(-1, 0)$ and sketch the solution through that point.

(d) (2 points) Plot the point $(0, 1)$ and sketch the solution through that point.

(e) (2 points) Plot the point $(1, 2)$ and sketch the solution through that point.

3. (33 points) Solve the first order differential equation.

(a) (11 points) $x' = \dfrac{t^2}{\cos(x)}$

Solution: This is a **separable** equation. Separate variables and integrate to get

$$\int \cos(x)\mathrm{d}x = \int t^2 \mathrm{d}t.$$

Calculating the integrals yields

$$\sin(x) = \frac{t^3}{3} + C.$$

Solving for x then gives us

$$x(t) = \arcsin\left(\frac{t^3}{3} + C\right).$$

(b) (11 points) $x' = te^{t^2}$

Solution: This is a function to be solved by **directly integrating**:

$$\begin{aligned} x(t) &= \int te^{t^2}\mathrm{d}t \\ &\overset{u=t^2,\ \frac{1}{2}\mathrm{d}u=t\mathrm{d}t}{=} \frac{1}{2}\int e^u \mathrm{d}u \\ &= \frac{1}{2}e^u + C \\ &= \frac{1}{2}e^{t^2} + C. \end{aligned}$$

(c) (11 points) $tx' + x = \dfrac{1}{t}$

Solution: This is a **linear equation**.

Solution 1: The left-hand side can immediately be written as $(xt)' = \dfrac{1}{t}$, which can be integrated (as shown below).

Solution 2: This equation is not in normal form $x' + p(t)x = q(t)$, so we first divide by t to put it in normal form to get

$$x' + \frac{1}{t}x = \frac{1}{t^2}. \tag{1}$$

Now we make integrating factor $\mu = e^{\left(\int p(t)\mathrm{d}t\right)}$ to get

$$\mu(t) = e^{\int \frac{1}{t}\mathrm{d}t} = e^{\ln(t)} = t.$$

Multiplying (1) by t gives the form $tx' + x = \dfrac{1}{t}$ which can be written as

$$(tx)' = \frac{1}{t}.$$

Integrating yields

$$xt = \int \frac{1}{t}\mathrm{d}t = \ln(t) + C.$$

Solving for x by dividing by t yields

$$x(t) = \frac{1}{t}\ln(t) + \frac{C}{t}.$$

4. (12 points) It can be shown that the second order linear homogeneous differential equation

$$x'' + x = 0$$

has general solution

$$x(t) = c_1\cos(t) + c_2\sin(t).$$

Use this fact and the method of undetermined coefficients to determine a solution to the second order **non**homogeneous differential equation

$$x'' + x = e^{7t}.$$

Solution: The solution will be of the form

$$x(t) = c_1\cos(t) + c_2\sin(t) + x_p(t),$$

for some function x_p. Make the guess $x_p(t) = Ae^{7t}$, consequently $x_p'(t) = 7Ae^{7t}$ and $x_p''(t) = 49Ae^{7t}$. Plugging these into the differential equation yields

$$49Ae^{7t} + Ae^{7t} = e^{7t}.$$

Since e^{7t} is always nonzero, we divide by e^{7t} to get

$$49A + A = 1,$$

i.e. $50A = 1$. Therefore $A = \frac{1}{50}$. Therefore solutions to the differential equation take the form

$$x(t) = c_1\cos(t) + c_2\sin(t) + \frac{1}{50}e^{7t}.$$

5. (35 points) Solve the given problem.

(a) (11 points) Solve the differential equation $x'' + 4x' + 4x = 0$.
Solution: Make guess $x(t) = e^{\lambda t}$, consequently $x'(t) = \lambda e^{\lambda t}$ and $x''(t) = \lambda^2 e^{\lambda t}$. Plugging these into the differential equation yields

$$\lambda^2 e^{\lambda t} + 4\lambda e^{\lambda t} + 4e^{\lambda t} = 0.$$

Since $e^{\lambda t}$ is always nonzero, divide it off to get

$$\lambda^2 + 4\lambda + 4 = 0.$$

The left-hand side factors yielding

$$(\lambda + 2)^2 = 0.$$

Therefore, $\lambda = -2$ – this is a double root. Therefore the solution is

$$x(t) = c_1 e^{-2t} + c_2 t e^{-2t}.$$

(b) (11 points) Solve the differential equation $x'' - 2x' + 3x = 0$.
Solution: Similar to (*a*), make guess $x(t) = e^{\lambda t}$ and plug in its derivatives into the differential equation, yielding

$$\lambda^2 - 2\lambda + 3 = 0.$$

The left-hand side does not factor, and so its roots may be found by using the quadratic formula:

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(3)}}{2} = \frac{2 \pm \sqrt{-8}}{2} = 1 \pm \frac{2\sqrt{2}i}{2} = 1 \pm \sqrt{2}i.$$

Therefore, the solution is

$$x(t) = c_1 e^t \cos(\sqrt{2}t) + c_2 e^t \sin(\sqrt{2}t).$$

(c) (13 points) Solve the initial value problem

$$\begin{cases} x'' + 4x = 0 \\ x(0) = 2, \quad x'(0) = 1. \end{cases}$$

Solution: Making guess $x(t) = e^{\lambda t}$ yields

$$\lambda^2 + 4 = 0.$$

Subtract 4 and take square root to get

$$\lambda = \sqrt{-4} = 2i.$$

Therefore, the solution is

$$x(t) = c_1 \cos(2t) + c_2 \sin(2t).$$

Now we need to find the initial conditions. First compute

$$x'(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t).$$

Now apply the initial conditions:

$$\begin{cases} 2 = & x(0) & = c_1 \cos(0) + c_2 \sin(0) \\ 1 = & x'(0) & = -2c_1 \sin(0) + 2c_2 \cos(0), \end{cases}$$

so

$$\begin{cases} 2 = c_1 + 0 \\ 1 = 2c_2, \end{cases}$$

meaning $c_1 = 2$ and $c_2 = \frac{1}{2}$. Therefore we have solution

$$x(t) = 2\cos(2t) + \frac{1}{2}\sin(2t).$$